Volume 70B
Number 4
Oct.–Dec. 1966

Mathematics and Mathematical Physics

U.S. DEPARTMENT OF COMMERCE

SECTION B

JOURNAL OF RESEARCH

NATIONAL BUREAU OF STANDARDS

Journal of Research

of the

National Bureau of Standards

B. Mathematics and Mathematical Physics

OCT.–DEC. • 1966

VOLUME • 70B

NUMBER • 4

Editor: Morris Newman

Associate Editors: John J. Sopka, F. W. Olver, and J. R. Edmonds

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1966

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Use of funds for printing this publication approved by the Director of the Bureau of the Budget (June 22, 1966).

Dedication of the New NBS Laboratories

On November 15, 1966, Secretary of Commerce John T. Connor dedicated the new ultra-modern facilities of the National Bureau of Standards at Gaithersburg, Md. This $120 million laboratory complex serves as the nerve center of the Nation's measurement and standardization activities. With three Institutes devoted to Basic Standards, Materials Research, and Applied Technology, the Bureau's new home has been designed to meet the needs of today and the challenges of the future in the physical sciences.

Dedication ceremonies were held in a courtyard between the Bureau's new library and one of the laboratory buildings. In the center of the courtyard is an apple tree descended from the tree under which Sir Isaac Newton was sitting when, according to the story, an apple fell on his head, leading him to conceive the law of universal gravitation.

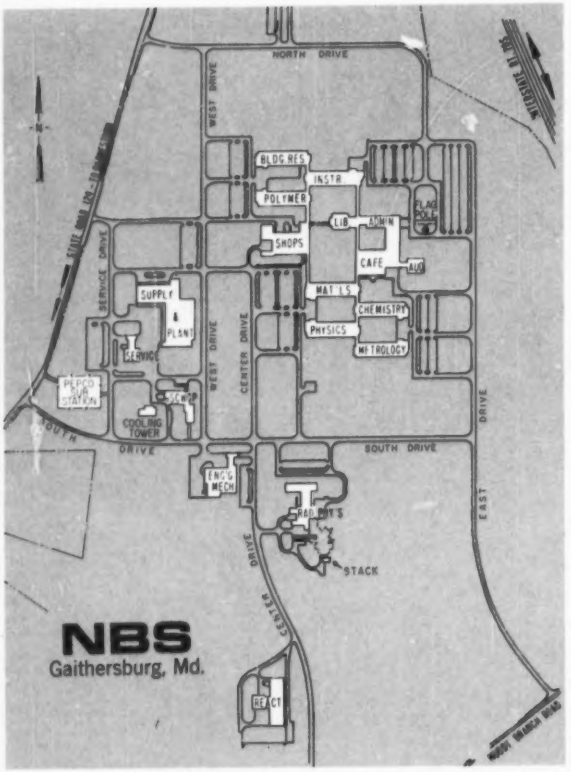
Surrounding this tree was an audience of 2500, including distinguished guests from government, industry, and the world scientific community. The Reverend Edward G. Latch, Chaplain of the House of Representatives, gave the invocation and the benediction. Introduced by Dr. A. V. Astin, Director of the National Bureau of Standards, Secretary Connor said:

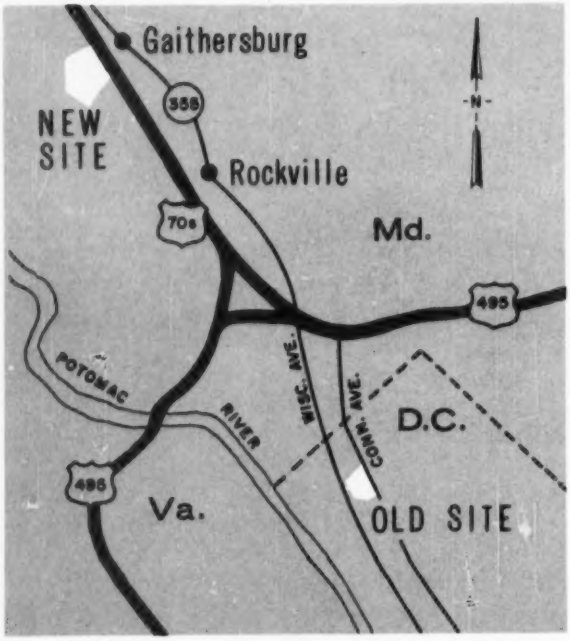
"It is no accident of history that the establishment of NBS in 1901 coincided with the beginning of America's transformation into the industrial giant of the twentieth century. . . . Today the demands made upon the Bureau and the nature of its program reflect the complexity of our atomic and space-age world."

The Secretary began his remarks with a message from President Johnson. The President noted that throughout its 65 years' existence the Bureau had advanced the frontiers of measurement in pace with the demands of science and industry; he extended his congratulations and best wishes for continuing achievement.

Secretary Connor stated that at age 65 the Bureau had significant accomplishments behind it and a future promising even greater contributions to the Nation.

Eleven stories high, the Administration Building at the National Bureau of Standards' new site near Gaithersburg, Md., towers above the other buildings in the laboratory complex. The building houses the Director and his staff as well as other NBS activities not requiring laboratory space.





The new NBS site is 20 miles northwest of the old Washington site, at the Darnestown exit of Interstate Highway 70S. It borders and is west of 70S and Gaithersburg, Md. The plan view shows the layout of existing buildings.

He noted that the new facilities are a symbol of the Bureau's future, but that the germ of future achievement lies in the members of the Bureau's staff.

In connection with the dedication, the Secretary sponsored a Symposium on Technology and World Trade which was held at the Bureau during the two days following the dedication. It featured as speakers experts from industry, government, and education of many countries, attracting an attendance of nearly 500.

Following the dedication ceremonies, the invited guests were given a tour of the installation. On Saturday, November 19, the general public was invited to visit the new facilities and to view over 100 laboratories and demonstrations. In an eight-hour period over 20,000 people roamed the site. Some of the things they saw were a 100-ft linear accelerator which produces one of the world's most intense electron beams; a recently completed high-flux research reactor; a 12-million pound capacity hydraulic testing machine; the NBS Museum containing historical instruments, equipment, standards, and materials pertaining to the Bureau's activities during its 65-year history; a vault containing the national measurement standards—or parts symbolizing those standards; a demonstration of glass blowing techniques used to make highly unusual, and precise apparatus; and many of the Bureau's laboratories in the fields of metrology, electricity, mechanics, heat, atomic physics, radiation physics, analytical chemistry, physical chemistry, polymers, metallurgy, and building research.

By the close of 1966, transfer of the Bureau's Washington (D. C.) laboratories to the Gaithersburg site will be largely completed and most of the 2700 staff members will be relocated in 15 major buildings at the site. Of about 300 persons remaining at the Washington site, most will be relocated in new special-purpose laboratories now under construction at Gaithersburg. In addition, the Bureau will continue to maintain a staff of about 600 in Boulder, Colo., for work in radio standards and cryogenics, and a staff of about 300 at the Clearinghouse for Federal Scientific and Technical Information at Springfield, Va., as well as small groups of employees at widely scattered field stations.

Although the Bureau is located 20 miles northwest of its old site, the mailing address remains: National Bureau of Standards, Washington, D. C. 20234. The new telephone number is (Area Code 301) 921–1000. To reach the Bureau by car, use the Darnestown exit on Interstate Highway 70S.

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

A Random Walk Model of Chain Polymer Adsorption at a Surface

III. Mean Square End-to-End Distance

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(September 13, 1966)

A 6-choice simple cubic lattice model of adsorption of an isolated polymer chain at a solution surface is investigated. The mean square components $\langle x^2(N)\rangle$ and $\langle z^2(N)\rangle$ of the end-to-end distance are computed as a function of the adsorption energy per monomer unit in the limit of a very long polymer chain. In the calculation, one end of the polymer chain consisting of N monomer units is constrained to lie in the surface; and $\langle x^2(N)\rangle$ and $\langle z^2(N)\rangle$ are, respectively, the mean square displacement of the free end of the chain parallel to the solution surface in one of the lattice directions and normal to the solution surface. The limiting value of $\langle x^2(N)\rangle/N$ as $N\to\infty$ is a continuous function of θ, the dimensionless adsorption energy per monomer unit, and is equal to 1/3 for $\theta \leq \ln(6/5)$ and $(1/2)\,[1+(1/4)\,(e^{\theta}-1)^{-1}]^{-1/2}$ for $\theta \geq \ln(6/5)$. The limiting value of $\langle z^2(N)\rangle/N$ as $N\to\infty$ is a discontinuous function of θ and is equal to 2/3 for $\theta < \ln(6/5)$, 1/3 for $\theta = \ln(6/5)$, and 0 for $\theta > \ln(6/5)$. The relation of these results to earlier investigations and the generalization of these results to other cubic lattice models is discussed.

Key Words: Adsorption, chain polymer, critical energy, generating function, lattice model, partition function, random walk.

1. Introduction

In this paper we extend the investigation of a random walk lattice model of polymer chain adsorption at a solution surface [1, 2][1] by computing the components of the mean square end-to-end distance in an adsorbed polymer chain. The random walk lattice model used is a generalization of the models introduced by Silberberg [3] and DiMarzio and McCrackin [4]. The physical properties of the model reflect the competition between (1) the energy gained by the polymer chain occupying as many surface sites as possible, and (2) the configurational entropy lost by the polymer chain remaining near the confining solution surface. In I and II the average fraction of monomer units which lie in the solution surface $\nu(\theta, N)$ was computed as a function of the reduced energy of adsorption per monomer unit $\theta=\epsilon/kT$. In the calculation one end of the polymer chain was constrained to lie in the surface, and the self-excluded volume of the polymer chain was neglected. In the limit in which the number of monomer units N is large, it was determined that there is a critical value of the reduced energy of adsorption per monomer unit θ_c such that for $\theta > \theta_c$, the molecule exists in an adsorbed state. For example, $\lim_{N\to\infty} \nu(\theta, N)$, the limiting average fraction of monomer units lying in the surface, is a positive constant independent of N indicating that a finite fraction of all monomer units lie in the solution surface. For $\theta < \theta_c$ this limiting fraction is zero.

The purpose of this paper is to calculate $\langle x^2(N)\rangle$, $\langle y^2(N)\rangle$, and $\langle z^2(N)\rangle$, the components of the mean square displacement of the free end of the polymer chain as a function of θ for $N \gg 1$ for the 6-choice simple cubic lattice model. It was shown in I that the mean distance of the free end of the chain from the solution surface $\langle z(N)\rangle$ has the following discontinuous form for the 6-choice

[1] Figures in brackets indicate the literature references at the end of this paper.

simple cubic lattice model

$$\lim_{N\to\infty} \langle z(N)\rangle/N^{1/2} = \begin{cases} (\pi/6)^{1/2}, & 0 \le \theta < \ln (6/5) \\ (2/3\pi)^{1/2}, & \theta = \ln (6/5) \\ 0, & \theta > \ln (6/5). \end{cases}$$

It is shown in section 4 that there is a corresponding discontinuity in $\lim_{N\to\infty} \langle z^2(N)\rangle/N$ whereas for the other components, $\lim_{N\to\infty} \langle x^2(N)\rangle/N = \lim_{N\to\infty} \langle y^2(N)\rangle/N$ is a continuous function of θ. The recurrence equations for the probability distribution of the free end of the polymer chain are formulated in section 2 and are solved formally in section 3. Explicit expressions for the second moments are evaluated in the limit $N\to\infty$ in section 4. The results are summarized in section 5 and the qualitative behavior for the 2-choice and 4-choice simple cubic models is deduced.

2. Recurrence Equations for the Probability Distribution of the Free End of the Polymer Chain

We consider a simple-cubic lattice model of the solution-surface system in which the solution surface corresponds to the x-y lattice plane through the point $z=0$. Successive lattice planes through $z=1, 2, \ldots$ represent the solution phase of the system. Polymer chain configurations in the solution correspond to paths generated in a random walk on the lattice between nearest-neighbor sites. The physical presence of the surface is introduced by considering only random walks in the lattice planes through $z=0, 1, 2, \ldots$ which never enter the lattice plane through $z=-1$. In the absence of a solution surface, all random walk configurations of a given length are equally likely. However, we are primarily interested in the influence of an adsorbing solution surface on the average conformation of a polymer chain. In our lattice model, all random walk paths of N steps with n steps lying in the surface layer $z=0$ have the same *a priori* probability. Relative to a random walk configuration of N steps with $n-1$ steps lying in the surface, the *a priori* probability of a walk with n steps in the surface layer is greater by the factor e^{θ} where $\theta=\epsilon/kT$ and ϵ is the adsorption energy of a monomer unit.

For convenience, we use a random walk terminology. Consider the problem of computing, for a random walk originating in the surface layer, the unnormalized or relative probability $P(x, y, z; N+1)$ that at the $N+1$th step the random walker is located at lattice point (x, y, z), where $z \ge 0$. The relative probability $P(x, y, z; N+1)$ is related to the relative probabilities at the Nth step by the relations

$$P(x, y, z; N+1) = \frac{1}{6}\{E_x^+ + E_x^- + E_y^+ + E_y^- + E_z^+ + E_z^-\}P(x, y, z; N), \quad z \ge 1 \tag{1}$$

and

$$P(x, y, 0; N+1) = \frac{1}{6} e^{\theta}\{E_x^+ + E_x^- + E_y^+ + E_y^- + E_z^+\}P(x, y, 0; N) \tag{2}$$

where $E_x^{\pm}$ are operators defined by the relation

$$E_x^{\pm} P(x, y, z; N) = P(x \pm 1, y, z; N). \tag{3}$$

The operators $E_y^{\pm}$ and $E_z^{\pm}$ have similar definitions. Equation (1) describes the relation between the relative probability of being at lattice site x, y, z with $z \ge 1$ at the $N+1$th step and the relative probabilities of being at neighboring sites at the Nth step. The factor e^{θ} in eq (2) accounts for the fact that relative to those configurations where $z \ge 1$ at the $N+1$th step, the relative probabilities for those configurations where $z=0$ at the $N+1$th step are greater by the factor e^{θ}. The absence of E_z^- in eq (2) is related to the fact that the random walker enters the $z=0$ layer only from one direction. Equations (1) and (2) will be solved in section 3 for the initial condition

$$P(x, y, z; 0) = \begin{cases} e^{\theta}, & x=0,\ y=0,\ z=0 \\ 0, & \text{for all other lattice points.} \end{cases} \tag{4}$$

From the solution, the mean square end-to-end distance after N steps can be computed from the expression

$$R_N^2 = \langle x^2(N)\rangle + \langle y^2(N)\rangle + \langle z^2(N)\rangle = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} (x^2+y^2+z^2) P(x, y, z; N)/Q(\theta, N) \tag{5}$$

where

$$Q(\theta, N) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} P(x, y, z; N). \tag{6}$$

3. Solution of Recurrence Equations

The recurrence eqs (1) and (2) can be solved for the starting condition eq (4) by introducing a generating function as in I and II. It is convenient first to rewrite eq (2) as

$$[1-(1-e^{\theta})]P(x, y, 0; N+1) = (1/6)[E_x^+ + E_x^- + E_y^+ + E_y^- + E_z^+]P(x, y, 0; N). \tag{7}$$

Multiply eqs (1) and (7) for $P(x, y, z; N+1)$ by $(2\pi)^{-3/2} \exp\{ix\xi + iy\eta + iz\zeta\}$ and sum over all integer values of x and y and all $z \geq 0$. The result is

$$-(1-e^{-\theta})(2\pi)^{-1/2} p(\xi, \eta; 0; N+1) + G(\xi, \eta, \zeta; N+1)$$

$$= \frac{1}{3}[\cos\xi + \cos\eta + \cos\zeta] G(\xi, \eta, \zeta; N) - \frac{1}{6} e^{-i\zeta} (2\pi)^{-1/2} p(\xi, \eta; 0; N) \tag{8}$$

where

$$G(\xi, \eta, \zeta; N) = (2\pi)^{-3/2} \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} P(x, y, z; N) \exp(ix\xi + iy\eta + iz\zeta) \tag{9}$$

and

$$p(\xi, \eta; z; N) = (2\pi)^{-1} \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} P(x, y, z; N) \exp(ix\xi + iy\eta). \tag{10}$$

Next multiply eq (8) by w^{N+1} and sum over all values of N to obtain

$$\Gamma(\xi, \eta, \zeta; w) - G(\xi, \eta, \zeta; 0) - (1-e^{-\theta})(2\pi)^{-1/2}[\rho(\xi, \eta; 0; w) - p(\xi, \eta; 0; 0)]$$

$$= \frac{1}{3} w[\cos\xi + \cos\eta + \cos\zeta] \Gamma(\xi, \eta, \zeta; w) - \frac{w}{6} e^{-i\zeta} (2\pi)^{-1/2} \rho(\xi, \eta; 0; w) \tag{11}$$

where

$$\Gamma(\xi, \eta, \zeta; w) = \sum_{N=0}^{\infty} w^N G(\xi, \eta, \zeta; N) \tag{12}$$

and

$$\rho(\xi, \eta; z; w) = \sum_{N=0}^{\infty} w^N p(\xi, \eta; z; N). \tag{13}$$

In eq (11) the quantities $G(\xi, \eta, \zeta; 0)$ and $p(\xi, \eta; 0; 0)$ are simply

$$G(\xi, \eta, \zeta; 0) = (2\pi)^{-3/2} e^{\theta} \tag{14}$$

and

$$p(\xi, \eta; 0; 0) = (2\pi)^{-1} e^{\theta}. \tag{15}$$

Substituting eq (14) and (15) in eq (11) and solving for $\Gamma(\xi, \eta, \zeta; w)$, one obtains

$$\Gamma(\xi, \eta, \zeta; w) = \frac{(2\pi)^{-3/2} + (2\pi)^{-1/2}\left[1 - e^{-\theta} - \frac{1}{6} w e^{-i\zeta}\right] p(\xi, \eta; 0; w)}{1 - \frac{1}{3} w[\cos \xi + \cos \eta + \cos \zeta]}. \tag{16}$$

Equation (16) is an implicit equation for $p(\xi, \eta; 0; w)$ because according to the definitions of $\Gamma(\xi, \eta, \zeta; w)$, $G(\xi, \eta, \zeta; N)$, $p(\xi, \eta; z; w)$, and $p(\xi, \eta; z; N)$ in eqs (9), (10), (12), and (13)

$$\Gamma(\xi, \eta, \zeta; w) = (2\pi)^{-3/2} \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} \sum_{N=0}^{\infty} P(x, y, z; N) w^N \exp(ix\xi + iy\eta + iz\zeta) \tag{17}$$

$$= (2\pi)^{-1/2} \sum_{z=0}^{\infty} e^{iz\zeta} p(\xi, \eta; z; w).$$

The following relation for determining $p(\xi, \eta; 0; w)$ can be obtained by multiplying eq (16) by $(2\pi)^{-1/2}$ and integrating with respect to ζ from $-\pi$ to π

$$p(\xi, \eta; 0; w) = (2\pi)^{-1} I_0(\xi, \eta; w) + \left[(1 - e^{-\theta}) I_0(\xi, \eta; w) - \frac{1}{6} w I_{-1}(\xi, \eta; w)\right] p(\xi, \eta; 0; w) \tag{18}$$

where

$$I_k(\xi, \eta; w) = (2\pi)^{-1} \int_{-\pi}^{\pi} \frac{e^{ik\zeta} d\zeta}{1 - w(\cos \xi + \cos \eta + \cos \zeta)/3}$$

$$= \left[1 - w(\cos \xi + \cos \eta)/3\right]^{-1} \left[1 - \left(\frac{w/3}{1 - w(\cos \xi + \cos \eta)/3}\right)^2\right]^{-1/2}$$

$$\times \left\{\frac{1 - \left[1 - \left(\frac{w/3}{1 - w(\cos \xi + \cos \eta)/3}\right)^2\right]^{1/2}}{\frac{w/3}{1 - w(\cos \xi + \cos \eta)/3}}\right\}^{|k|}. \tag{19}$$

Solving eq (18) for $\rho(\xi, \eta; 0; w)$ and substituting the result in eq (16), we obtain an explicit solution of the recurrence eqs (1) and (2) for the initial condition (4)

$$\Gamma(\xi, \eta, \zeta; w) = \frac{(2\pi)^{-3/2}}{1 - w(\cos \xi + \cos \eta + \cos \zeta)/3} \left\{ \frac{1 + (w/6)[I_1(\xi, \eta; w) - e^{-i\zeta} I_0(\xi, \eta; w)]}{1 + (w/6) I_1(\xi, \eta; w) - (1 - e^{-\theta}) I_0(\xi, \eta; w)} \right\}. \quad (20)$$

The mean square end-to-end distance (5) can be expressed in terms of the generating function $\Gamma(\xi, \eta, \zeta; w)$. First note that the normalizing sum $Q(\theta, N)$, defined in eq (6), is proportional to the coefficient of w^N in the expansion of $\Gamma(0, 0, 0; w)$ [see eq (17)]

$$\Gamma(0, 0, 0; w) = (2\pi)^{-3/2} \sum_{N=0}^{\infty} w^N \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} P(x, y, z; N) = (2\pi)^{-3/2} \sum_{N=0} w^N Q(\theta, N) \quad (21)$$

Using Cauchy's formula and eq (20), we obtain the following explicit expression for $Q(\theta, N)$

$$Q(\theta, N) = \frac{(2\pi)^{3/2}}{2\pi i} \int_{C_0} \Gamma(0, 0, 0; w) \frac{dw}{w^{N+1}} = \frac{1}{2\pi i} \int_{C_0} \frac{1 + [(1 - w/3)/(1 - w)]^{1/2}}{[(1 - w)(1 - w/3)]^{1/2} - 1 - (2w/3) + 2e^{-\theta}} \frac{dw}{w^{N+1}}, \quad (22)$$

where C_0 is a counterclockwise contour of integration around $w = 0$. It also follows from eq (17) and the symmetry in the x and y coordinates that the numerator in eq (5) is proportional to the coefficient of w^N in

$$-\left[2\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}\right] \Gamma(\xi, 0, \zeta; w)\Big|_{\xi=0, \zeta=0} = (2\pi)^{-3/2} \sum_{N=0}^{\infty} w^N \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} \sum_{z=0}^{\infty} (x^2 + y^2 + z^2) P(x, y, z; N). \quad (23)$$

The explicit expression for the numerator in eq (5) is

$$2\mathcal{I}_x(N) + \mathcal{I}_z(N) = \frac{-(2\pi)^{3/2}}{2\pi i} \int_{C_0} \left[2\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}\right] \Gamma(\xi, 0, \zeta; w) \Big|_{\xi=0, \zeta=0} \frac{dw}{w^{N+1}}, \quad (24)$$

or

$$\mathcal{I}_x(N) = \frac{1}{2\pi i} \int_{C_0} \left\{ \frac{(w/3)^2}{(1 - w/3)^{1/2} (1 - w)^{3/2}} \frac{1}{D(\theta, w)} + \frac{w/3}{1 - w} \frac{1}{D^2(\theta, w)} \right.$$

$$\left. \times \left[1 + \left(\frac{1 - w}{1 - w/3}\right)^{1/2}\right] [1 - (2w/3) + (1 - w/3)^{1/2} (1 - w)^{1/2}] \right\} \frac{dw}{w^{N+1}} \quad (25)$$

and

$$\mathcal{I}_z(N) = \frac{1}{2\pi i} \int_{C_0} \frac{1/3}{D(\theta, w)} \left(\frac{(1 - w/3)^{1/2}}{(1 - w)^{3/2}}\right) \frac{dw}{w^N} \quad (26)$$

where

$$D(\theta, w) = [(1-w)(1-w/3)]^{1/2} - 1 - (2w/3) + 2e^{-\theta}. \tag{27}$$

In eq (24) $\mathcal{I}_x(N)$ and $\mathcal{I}_z(N)$ are, respectively, the unnormalized mean square x and z components of the position of the free end of the chain. The explicit expressions (25) and (26) for $\mathcal{I}_x(N)$ and $\mathcal{I}_z(N)$ are evaluated in the limit $N \to \infty$ in the next section.

4. Evaluation of R_N^2 for $N \gg 1$

We now consider the problem of obtaining asymptotic formulas in the limit $N \gg 1$ for the x and z components of R_N^2 in eq (5). The explicit expressions which must be evaluated are

$$\langle x^2(N) \rangle = \mathcal{I}_x(N)/Q(\theta, N)$$

and

$$\langle z^2(N) \rangle = \mathcal{I}_z(N)/Q(\theta, N)$$

where $\mathcal{I}_x(N)$, $\mathcal{I}_z(N)$, and $Q(\theta, N)$ are given in eqs (25), (26), and (22). The details of the evaluation of these contour integrals in the limit $N \gg 1$ are straightforward but lengthy and depend as in I upon whether $\theta > \theta_c$, $\theta = \theta_c$, or $\theta < \theta_c$ where $\theta_c = \ln (6/5)$. All three integrands have branch points at $w = 1$ and $w = 3$. As in I, we introduce a cut in the w-plane between these branch points as shown in figure 1. In addition to the branch points, when $\infty > \theta > \theta_c$, there is a zero of $D(\theta, w)$ on the real axis between $w = 0$ and $w = 1$. Details are given in I where it is shown that the expression for this zero is

$$w_+ = 6\{-2(1-e^{-\theta}) + [4(1-e^{-\theta})^2 + e^{-\theta}(1-e^{-\theta})]^{1/2}\}. \tag{28}$$

It is also shown that w_+ approaches 0 as $\theta \to \infty$ and it approaches 1 as $\theta \to \theta_c = \ln (6/5)$. Thus when $\theta = \theta_c$, the pole of the integrands at w_+ coincides with the branch point at $w = 1$. In the remainder of this section we obtain asymptotic formulas for the contour integrals (25), (26), and (22) in the limit $N \gg 1$ for the three cases $\theta > \theta_c$, $\theta = \theta_c$, and $\theta < \theta_c$.

$\theta > \theta_c$. For $\theta > \theta_c$ the contour integrals $\mathcal{I}_x(N)$, $\mathcal{I}_z(N)$, and $Q(\theta, N)$ all satisfy the relation

$$\frac{1}{2\pi i}\int_{C_0} R(w)dw = -\frac{1}{2\pi i}\int_{C_1} R(w)dw + \frac{1}{2\pi i}\int_{C_2} R(w)dw \tag{29}$$

where the integral on the left-hand side of eq (29) stands for $\mathcal{I}_x(N)$, $\mathcal{I}_z(N)$, or $Q(\theta, N)$ and the contours are shown in figure 1. In the limit $N \gg 1$, the second integral on the right-hand side of eq (29) is negligible compared to the first integral. Therefore, the asymptotic values of $\mathcal{I}_x(N)$, $\mathcal{I}_z(N)$, and $Q(\theta, N)$ are given by the residues of the appropriate integrand at w_+. The values are

$$Q(\theta, N) \cong -w_+^{-(N+1)}\{1 + [(1-w_+/3)/(1-w_+)]^{1/2}\}/D'(\theta, w_+) \tag{30}$$

$$\mathcal{I}_z(N) \cong -\frac{1}{3}w_+^{-N}(1-w_+/3)^{1/2}/(1-w_+)^{3/2}D'(\theta, w_+) \tag{31}$$

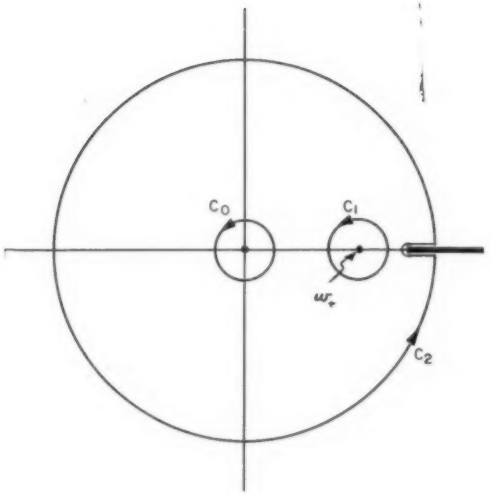

FIGURE 1. *Integration contours* C_0, C_1, *and* C_2 *in the complex w-plane.*
The contour C_1 is indented around the left end of the cut terminating at $w = 1$.

and

$$\mathscr{I}_x(N) \cong \frac{N}{w_+^{N+2}} \frac{w_+/3}{1-w_+} \left[1+\left(\frac{1-w_+}{1-w_+/3}\right)^{1/2}\right]$$

$$\times [1-(2w_+/3)+(1-w_+/3)^{1/2}(1-w_+)^{1/2}]\,[D'(\theta, w_+)]^{-2} \qquad (32)$$

where $D'(\theta, w_+)$ denotes $(d/dw)D(\theta, w)|_{w=w_+}$ and where terms proportional to N^0 in eq (32) have been neglected. Equation (30) is a special case of eq (37) in I. Combining eqs (30) and (31), we obtain for the mean square z-component of $\langle R_N^2 \rangle$

$$\langle z^2(N)\rangle \cong \frac{w_+}{3} \frac{(1-w_+/3)^{1/2}}{(1-w_+)^{3/2}} \Big/ \left[1+\left(\frac{1-w_+/3}{1-w_+}\right)^{1/2}\right] \cong \frac{w_+}{2}\left(\frac{1-w_+/3}{1-w_+}\right)^{1/2}\left[\left(\frac{1-w_+/3}{1-w_+}\right)^{1/2}-1\right]. \qquad (33)$$

Thus, it is seen that if $\theta > \ln(6/5)$, then $w_+ < 1$; and the limiting value of $\langle z^2(N)\rangle/N$ is

$$\lim_{N\to\infty} \langle z^2(N)\rangle/N = 0. \qquad (34)$$

Combining eqs (30) and (32), we obtain for the mean square x-component of $\langle R_N^2 \rangle$

$$\langle x^2(N)\rangle \cong -\frac{N}{3}\left[1+\frac{1-(2w_+/3)}{(1-w_+)^{1/2}(1-w_+/3)^{1/2}}\right][D'(\theta, w_+)]^{-1} \cong \frac{N}{2}\left[1+\frac{1}{4}\frac{e^{-\theta}}{1-e^{-\theta}}\right]^{-1/2}. \qquad (35)$$

In obtaining eq (35) we used the relation

$$D'(\theta, w_+) = \frac{1}{3}\left[\frac{-2+w_+}{(1-w_+)^{1/2}\ (1-w_+/3)^{1/2}} - 2\right] = -\frac{1}{3}\left[\frac{4(1-e^{-\theta})+w_+/3}{1+(2w_+/3)-2e^{-\theta}}\right] \tag{36}$$

and eq (28) for w_+. The limiting value of $\langle x^2(N)\rangle/N$ is

$$\lim_{N\to\infty}\ \langle x^2(N)\rangle/N = \frac{1}{2}\left[1+\frac{1}{4}\frac{e^{-\theta}}{1-e^{-\theta}}\right]^{-1/2}. \tag{37}$$

$\theta = \theta_c$. For $\theta = \theta_c = \ln(6/5)$ the factor $[D(\theta, w)]^{-1}$ appearing in the integrands of $Q(\theta, N)$, $\mathscr{I}_x(N)$, and $\mathscr{I}_z(N)$ can be written as

$$[D(\theta_c, w)]^{-1} = \frac{9}{5+w}\left[\left(\frac{1-w/3}{1-w}\right)^{1/2} - \frac{2}{3}\right]. \tag{38}$$

When $[D(\theta_c, w)]^{-1}$ is replaced by eq (38), the expressions for $Q(\theta, N)$, $\mathscr{I}_z(N)$, and $\mathscr{I}_x(N)$ become

$$Q(\theta, N) = \frac{1}{2\pi i}\int_{C_0}\frac{9}{5+w}\left[\frac{1-w/3}{1-w} + \frac{1}{3}\left(\frac{1-w/3}{1-w}\right)^{1/2} - \frac{2}{3}\right]\frac{dw}{w^{N+1}}. \tag{39}$$

$$\mathscr{I}_z(N) = \frac{1}{2\pi i}\int_{C_0}\frac{3}{5+w}\left[\frac{1-w/3}{(1-w)^2} - \frac{2}{3}\frac{(1-w/3)^{1/2}}{(1-w)^{3/2}}\right]\frac{dw}{w^N} \tag{40}$$

and

$$\mathscr{I}_x(N) = \frac{1}{2\pi i}\int_{C_0}\left\{\frac{w^2}{5+w}\left[\frac{1}{(1-w)^2} - \frac{2/3}{(1-w/3)^{1/2}(1-w)^{3/2}}\right]\right.$$

$$+\frac{27w}{(5+w)^2}\left[\frac{1-w/3}{(1-w)^2} - \frac{4}{3}\frac{(1-w/3)^{1/2}}{(1-w)^{3/2}} + \frac{4/9}{1-w}\right]$$

$$\left.\times\left[1+\left(\frac{1-w}{1-w/3}\right)^{1/2}\right][1-(2w/3)+(1-w/3)^{1/2}(1-w)^{1/2}]\right\}\frac{dw}{w^{N+1}}. \tag{41}$$

The three contour integrals now satisfy the relation

$$\frac{1}{2\pi i}\int_{C_0} R(w)dw = \frac{1}{2\pi i}\int_{C_2} R(w)dw. \tag{42}$$

It is readily verified that in each case the dominant contribution to the integrals (40) to (42) around the contour C_2 comes from the most singular terms in $(1-w)$. In (40) to (42) these singularities are simple poles. Evaluating the residues at these poles and retaining terms proportional to the highest power of N only, one obtains

$$Q(\theta_c, N) \cong 1, \tag{43}$$

$$\mathscr{I}_z(N) \cong N/3, \tag{44}$$

and

$$\mathscr{I}_x(N) \cong N/3. \tag{45}$$

The limiting values of $\langle z^2(N)\rangle/N$ and $\langle x^2(N)\rangle/N$ are

$$\lim_{N\to\infty} \langle z^2(N)\rangle/N = \frac{1}{3} \tag{46}$$

and

$$\lim_{N\to\infty} \langle x^2(N)\rangle/N = \frac{1}{3}. \tag{47}$$

$\theta < \theta_c$. For $\theta < \theta_c$, the contour integrals (22), (26), and (25) satisfy the relation (42). The dominant contribution to the contour integrals for $Q(\theta, N)$, $\mathscr{I}_z(N)$, and $\mathscr{I}_x(N)$ around C_2 comes from the most singular terms in $(1-w)$ in the immediate vicinity of $w=1$. The expressions for these dominant contributions are

$$Q(\theta, N) \cong \frac{1}{2\pi i}\int_Q^R \left(\frac{1-w/3}{1-w}\right)^{1/2} \frac{1}{D(\theta, w)} \frac{dw}{w^{N+1}}, \tag{48}$$

$$\mathscr{I}_z(N) \cong \frac{1}{2\pi i}\int_Q^R \frac{(1-w/3)^{1/2}}{(1-w)^{3/2}} \frac{1/3}{D(\theta, w)} \frac{dw}{w^N}, \tag{49}$$

and

$$\mathscr{I}_x(N) \cong \frac{1}{2\pi i}\int_Q^R \frac{1/9}{(1-w/3)^{1/2}(1-w)^{3/2}} \left(\frac{1}{D(\theta, w)}\right) \frac{dw}{w^{N+1}} \tag{50}$$

where the points Q and R are shown in figure 1. It is possible to avoid the explicit evaluation of the contour integrals in (48) to (50) because we are interested only in obtaining asymptotic formulas for $\mathscr{I}_x(N)/Q(\theta, N)$ and $\mathscr{I}_z(N)/Q(\theta, N)$. To demonstrate this fact, integrate by parts in (50) to reduce the order of the singularity in $(1-w)$. The result for $\mathscr{I}_x(N)$ is

$$\mathscr{I}_x(N) \cong \frac{1}{2\pi i}\int_Q^R \frac{2N/9}{(1-w)^{1/2}} \frac{1}{(1-w/3)^{1/2}D(\theta, w)} \frac{dw}{w^N} \tag{51}$$

where we have neglected contributions of lower order in N. We also remark that the singularities in $1-w$ in (49) and (50) are identical, as are the singularities in $(1-w)$ in (48) and (51). Consequently, the asymptotic formula for $\mathscr{I}_z(N)$ can be deduced from that of $\mathscr{I}_x(N)$ by multiplying $\mathscr{I}_x(N)$ by the factor

$$\lim_{w\to 1} \; 3(1-w/3) = 2;$$

and the asymptotic formula for $\mathscr{I}_x(N)$ can be deduced from $Q(\theta, N)$ by multiplying $Q(\theta, N)$ by the factor

$$\lim_{w \to 1} \frac{2N}{9} \frac{1}{1-w/3} = \frac{N}{3}.$$

Thus we have the relations

$$\mathscr{I}_z(N) = 2\mathscr{I}_x(N)$$

and

$$\mathscr{I}_x(N) = NQ(\theta, N)/3 \tag{52}$$

for $N \gg 1$. The limiting values of $\langle x^2(N)\rangle/N$ and $\langle z^2(N)\rangle/N$ are then

$$\lim_{N \to \infty} \langle x^2(N)\rangle/N = 1/3 \tag{53}$$

and

$$\lim_{N \to \infty} \langle z^2(N)\rangle/N = 2/3. \tag{54}$$

5. Summary and Remarks

The limiting mean square components of the end-to-end displacement parallel to the solution surface are equal and have the value

$$\lim_{N \to \infty} \langle x^2(N)\rangle/N = \lim_{N \to \infty} \langle y^2(N)\rangle/N = \begin{cases} \frac{1}{3}, & 0 \leq \theta < \ln(6/5) \\ \frac{1}{3}, & \theta = \ln(6/5) \\ \frac{1}{2}\left[1 + \frac{1}{4}(e^{\theta} - 1)^{-1}\right]^{-1/2}, & \theta > \ln(6/5). \end{cases} \tag{55}$$

Thus $\lim_{N \to \infty} \langle x^2(N)\rangle/N$ is a continuous function of the reduced adsorption energy per monomer unit which increases monotonically from ⅓ to ½ in the θ-interval $[\ln(6/5), \infty)$. The limiting mean square component of the end-to-end displacement normal to the solution surface is

$$\lim_{N \to \infty} \langle z^2(N)\rangle/N = \begin{cases} \frac{2}{3}, & 0 \leq \theta < \ln(6/5) \\ \frac{1}{3}, & \theta = \ln(6/5) \\ 0, & \theta > \ln(6/5). \end{cases} \tag{56}$$

It is seen in (55) and (56) that at $\theta = \theta_c$ the limiting mean square components of the end-to-end distance are equal to 1/3, the value which would be obtained in the absence of the solution surface. This behavior at the critical energy was deduced by DiMarzio and McCrackin [4]. DiMarzio [5] has obtained our result in the special case $\theta = 0$, but for a general lattice model. In this case ($\theta = 0$), the mean square normal component is twice the mean square lateral components which are in turn equal to the value obtained in the absence of the solution surface. It is seen in (55) and (56) that as $\theta \to \infty$ and the molecule becomes confined to the solution surface, the mean square

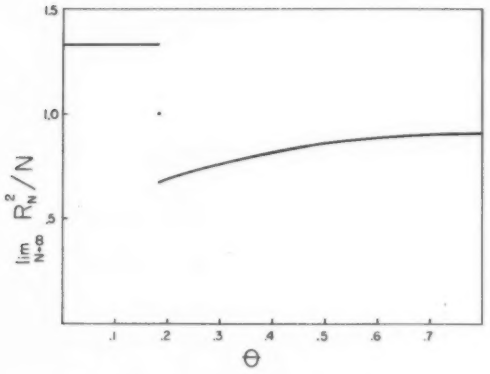

FIGURE 2. *Limiting form of the mean square end-to-end distance per step versus the adsorption energy θ.*

TABLE 1

Values of mean square components of the end-to-end displacement in the 6-choice, 4-choice, and 2-choice simple cubic lattice models

	$\langle x^2(N)\rangle/N$	$\langle z^2(N)\rangle/N$
$0 \le \theta < \theta_c$	1/3	2/3
$\theta = \theta_c$	1/3	1/3
$\theta = \infty$	1/2	0

lateral components approach the value which would be obtained for a random walk on a 4-choice planar square lattice. Combining the limiting forms for the mean square components of the end-to-end distance, we obtain finally for the mean square end-to-end distance per step

$$\lim_{N\to\infty} R_N^2/N = \begin{cases} \dfrac{4}{3}, & 0 \le \theta < \ln (6/5) \\ 1, & \theta = \ln (6/5) \\ \left[1 + \dfrac{1}{4}(e^{\theta} - 1)^{-1}\right]^{-1/2}, & \theta > \ln (6/5). \end{cases} \tag{57}$$

This function is plotted versus θ in figure 2.

We now consider the question of determining qualitatively the components of the mean square displacement in the 4-choice and the 2-choice simple cubic lattice models. These models have been studied [1, 2] and the critical energies $\theta_c = \ln (5^{1/2} - 1)$ and $(1/3) \ln 2$, respectively, have been obtained. We expect that analogous to the result in the 6-choice simple cubic lattice model, the mean square components of the end-to-end distance in the adsorption energy range $0 < \theta < \theta_c$ are equal to the values obtained at $\theta = 0$ by DiMarzio [5]. We also expect that at $\theta = \theta_c$, the mean square components of the end-to-end distance are equal to their free space values. As $\theta \to \infty$, the mean square components parallel to the surface approach the values for the corresponding planar lattice. In table 1 we list the limiting values of the mean square components of the end-to-end distance which are obtained on the basis of the foregoing arguments and note that values listed are the same in the 2-, 4-, and 6-choice simple cubic lattice models. Only the critical energies and the rates of approach to the $\theta = \infty$ values are different. The results for the 4-choice simple cubic lattice model are useful as reference values in McCrackin's Monte Carlo calculations of R_N^2 taking into account the self-excluded volume of the polymer chain [6].

6. References

[1] R. J. Rubin, J. Chem. Phys. **43,** 2392 (1965), referred to as I.
[2] R. J. Rubin, J. Res. NBS **69B,** 301 (1965), referred to as II.
[3] A. Silberberg, J. Phys. Chem. **66,** 1872 (1962).
[4] E. A. DiMarzio and F. L. McCrackin, J. Chem. Phys. **43,** 539 (1965).
[5] E. A. DiMarzio, J. Chem. Phys. **42,** 2101 (1965).
[6] F. L. McCrackin, Bull. Am. Phys. Soc. **10,** 380 (1965), manuscript submitted to J. Chem. Phys.

(Paper 70B4–185)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

Abscissas and Weights for Gaussian Quadrature for $N = 2$ to 100, and $N = 125$, 150, 175, and 200[1]

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(January 27, 1966)

The abscissas and weights for Gaussian Quadrature of order $N = 2$ to 100, and $N = 125$, 150, 175, and 200 are given. The abscissas are given to twenty-four places and the error is estimated to be no more than 1 unit in the last place. The weights are given to twenty-three places and the error is estimated to be no more than 1 unit in the last place.

Key Words: Gaussian quadrature, integral equations, numerical integration, zeros of Legendre polynomials.

1. Introduction

The advent of high speed digital computers has made the use of Gaussian Quadrature formulae of very high order a practical procedure. It can be programmed with a few instructions and the awkward abscissas are easily handled by the computer. It is a familiar technique which usually converges rapidly, and since the whole range of integration can be covered at once instead of making many subdivisions, it will usually be very efficient.

In addition to its use in numerical integration Gaussian Quadrature can be used in the numerical solution of integral equations and also in the evaluation of functions that can be written as integrals. The last four sets of very high order were computed [1][3] since "exact values of these quantities are also interesting in view of certain unsettled theoretical conjectures that have been made about distributions of the weights and abscissas." It was reported by Davis and Rabinowitz [2] that there was a "brisk demand" for the sets $N = 64$, 80, and 96 even though they had not been published and there existed some doubt about their accuracy.

Higher order Gaussian Quadratures were found to be very efficient in the calculation of the inductance of rectangular conductors such as strip transmission lines at various frequencies [3]. It was found that if the width of the outer conductors were divided into n equal parts and the width of the inner conductors according to a Gaussian distribution, then the rate of convergence of the inductance function (with respect to n the number of subdivisions) was greatly improved. In order to obtain the limiting value of inductance (for an infinite number of subdivisions) a formula of the form $L_\infty = L_n + an^{-\alpha}$ was used for sufficiently large n. This equation is most easily solved for the unknowns L_∞, a, and α, if the inductance function L_n is calculated for four different n chosen such that $N_1/N_2 = N_3/N_4$. Therefore, specific high order Gaussian Quadratures were needed.

The only high order Gaussian Quadratures that were found in the literature were those mentioned in the references. A. H. Stroud is working on tables for $N = 2$, 64, 96, 168, 256, 384, and 512 but he has not published them at the present time [4].

2. Method

The Gaussian Quadrature formula is given by

$$\int_{-1}^{1} F(X)dx \approx \sum_{k=1}^{n} H_k F(X_k), \qquad (1)$$

[1] The complete tables, 77 pages, have recently been published as NBS Monograph 98, Abscissas and Weights for Gaussian Quadrature For $N = 2$ to 100, and $N = 125$, 150, 175, and 200. This monograph is available from the Superintendent of Documents, Government Printing Office, Washington, D.C. 20402. Price 55 cents.

[2] Radio Standards Engineering, NBS Laboratory, Boulder, Colo.

[3] Figures in brackets indicate the literature references at the end of this paper.

where the weighing coefficients H_k are given by

$$H_k = \frac{2}{nP_{n-1}(X_k)P'_n(X_k)}; \tag{2}$$

$P_n(X)$ is Legendre's polynomial of degree n, $P'_n(X)$ is the first derivative of $P_n(X)$, and the $X_{k's}$ are the zeros of $P_n(X)$ [5].

From the equations

$$(1-X^2)P'_n(X) = nXP_n(X) + nP_{n-1}(X)$$

$$= (n+1)XP_n(X) - (n+1)P_{n+1}(X), \tag{3}$$

and with X_k a zero of $P_n(X)$ in (3), the following relations can be obtained:

$$(1-X_k^2)P'_n(X_k) = nP_{n-1}(X_k) = -(n+1)P_{n+1}(X_k). \tag{4}$$

Equation (2) can now be expressed as [6]

$$H_k = \frac{2}{(1-X_k^2)[P'_n(X_k)]^2} = \frac{2(1-X_k^2)}{(n+1)^2[P_{n+1}(X_k)]^2}. \tag{5}$$

Since $P_n(X)$ is a symmetric function of X for even n and skew symmetric for odd n, only the positive roots between 0 and 1 have to be calculated [5]. The weights H_k will be the same for positive and negative X's of the same magnitude.

The roots of the Legendre Polynomials and their corresponding weighing factors were computed as follows. Upper and lower bounds were obtained for the kth root of $P_n(X)$, namely $X_{n,k}$, from the fact that the roots of $P_{n-1}(X)$ separate the roots of $P_n(X)$:

$$\begin{aligned} 1 < X_{n,1} < X_{n-1,1} \qquad & \text{For } k=1 \\ X_{n-1,k-1} < X_{n,k} < X_{n-1,k} \qquad & \text{For } k>1. \end{aligned} \tag{6}$$

When n is odd, $X=0$ is also a root of $P_n(X)$.

Setting the lower limit equal to a and the upper limit equal to b, the method of false position can be used to obtain an initial approximation c:

$$C = a + \frac{F(a)}{F(a)-F(b)}(b-a) = a + \frac{P_n(a)}{P_n(a)-P_n(b)}(b-a). \tag{7}$$

In order to calculate $P_n(X)$ and $P_{n+1}(X)$ the following recurrence relation was used:

$$P_{n+1}(X) = \frac{2n+1}{n+1}XP_n(X) - \frac{n}{n+1}P_{n+1}(X). \tag{8}$$

After an approximation has been obtained, the root $X_{n,k}$ can be obtained much faster by using the Newton-Raphson technique which converges quadrically [6]:

$$X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)}, \tag{9}$$

where X_{n+1} is the next approximation to the root X_n. The value of $F(X_n)$ can be obtained from (4).

The last four sets ($n=125$, etc.) required a little different approach. The first root, $X_{n,1}$, was obtained for all n between 101 and 200. The fact that the roots of $P_n(X)$ became monotonically increasingly further apart was then used to locate succeeding roots:

$$1 - X_{n,1} < X_{n,1} - X_{n,2} < \ldots < X_{n,k} - X_{n,k+1}. \tag{10}$$

In order to find the $X_{n,k}$ root, the quantity $a = X_{n,k-1} - X_{n,k-2}$ was found. Then b was set equal to $X_{n,k-1} - a$ and a equal to $b-a$. The polynomials P_n(a) and P_n(b) were examined to see if they were of opposite sign. If so, the upper and lower bounds were found and the root was calculated as before. If they had the same sign, then a and b were both decreased by a and the above process repeated until P_n (a) and P_n (b) did indeed have opposite signs.

3. Errors

The calculations were done with a double precision computer routine. The routine was accurate to 1.9×10^{25} (or 2^{84}).

In order to check the accuracy of the abscissas the formula $\sum_1^n X_k^2 = \frac{n(n-1)}{2n-1}$ was used. The largest error found was 5 units in the 24th decimal place and the next largest was 2 units in the 24th place. More than half of the checks had zero error in the 24th place. Squaring a number that has an error in the kth place gives an error in the kth place that is approximately twice that of the original error. When checking the tables in references [1], [2], and [5] only one error in the abscissas was found. This was in reference [5] where an error of 0.5 units in the 21st place was discovered. The calculations had been carried out to 7.6×10^{22} and the abscissas given to 21 places. The 21st and 23d places are 851 but the value given in the table was 8 in the 21st place instead of being rounded up to 9.

In this report the abscissas have been given to 24 places and it has been concluded that there is an error of no more than 1 unit in the last place. It is also concluded that this occurrence will be very rare. There will be further comments on the abscissas after the weights have been examined.

The weights were checked by summing them between zero and one and comparing the results to unity. The largest error found was 1.48 units in the 23d place. The previously mentioned references were checked for errors in the weights and no error larger than 1 unit in the last place was found through $n=48$. In reference [5] a table of errors obtained by summing the weights in references [1] and [2] is given. Even though the error in reference [1] reaches 11 units in the last place, there is no error in any individual weight of more than 1 unit. It is interesting

to note that the errors in [1] are always 1 unit too small and are caused by not rounding up. It was only when the first omitted digit was 9 that the rounding was done correctly and then only half the time.

It was found that the errors in the higher order tables of reference [5] (up to $n=64$) were larger than expected. The sums of some weights were in error 4 units in the last place, but almost all the error was concentrated in the first or second weight. For example, the value of the first weight for $n=64$ was found to be 4.7 units too small in the last place and the sum of the weights was 4 units too small.

Since the preceding observation questioned the accuracy of the 23d place of the first two weights of higher n in this report, the following analysis was made. It was assumed that the errors would follow the same behavior pattern in double precision as in single precision. Starting at $n=50$, all the higher orders were calculated in single precision and the sum of the weights obtained. Single precision was 6.87×10^{10} (or 2^{36}) which is less than 11 significant digits. It was assumed that there were 11 good digits and the error in the 9th place was examined. For n up to 100 the largest error in the sum was 3.0 units in the 9th place and 2.2 units for an individual weight. The largest error in the last four sets was 3.9 units in the 9th place of the sum but only 2.3 units for any individual term. It was observed that if the error in the sum was 1 unit or more in the 9th place, it was concentrated in the first two weights.

These results tend to show that the errors do follow a similar pattern and give an error bound on the weights. It is to be remembered that in single precision there were *less* than 11 significant digits and the error in the 9th place was examined while in double precision there were *more* than 25 significant digits and the error in the 23d place was examined. Hence, the error in the 9th place in single precision should be greater than the error in the 23d place in double precision. It was concluded, on the basis of the preceding discussion and the fact that the largest error in the sum in double precision was 1.48 units in the 23d place, that the individual weights were accurate to within 1 unit in the 23d place.

The sum of the squares of the abscissas were checked and the largest error was 4 units in the 10th place. This was a better check than the double precision because of a "digit" problem. The sum of the checks reaches a value of approximately 50 for higher n and hence it is impossible to obtain the sum accurately to more than 23 decimal places. When the single precision was used, the squaring and summing were done in double precision and hence the error in the 10th decimal place could be examined. It has been mentioned previously that squaring tends to double the error and also that there are less than 11 good digits in single precision while in double precision there are more than 25 good digits. Hence, the error is no more than 1 unit in the 24th place.

The three values of n given in reference [2] were checked and were accurate to within 1 unit in the last place. This is better than one would expect from the above analysis since that author was only calculating his abscissas to an accuracy of 5.1×10^{-22}, which is approximately 1 digit less than reference [1]. It is believed the better accuracy was obtained because of the rapid rate of convergence of the Newton-Raphson method and because triple precision was used. The convergence rate was in fact observed to be approximately "squared"; the errors were approximately 10^{-4}, 10^{-8}, 10^{-16} for successive iterations. Hence, when the error condition was satisfied, the roots were probably more accurate than expected and therefore the weights were more accurate. It is believed that the larger error of the first two weights is caused by the $1-X^2$ term in the numerator of the expression for the weights.

4. Table Errors

After the final form of the manuscript had been prepared, the abscissas and weights were punched onto cards from the manuscript. These values were then subjected to the same numerical checks as before and the errors checked to insure they were the same as before. The values were also compared against the values that were punched onto cards during the computation.

Tables. Examples of the tables for $N=2$ to 100, and $N=125$, 150, 175, and 200 follow. The complete set of tables will be published in a NBS monograph by Carl H. Love.

234-094 O-67—2

N = 70

ABSCISSAS	WEIGHTS
0.99941 82859 73575 84205 6743	0.00149 27212 88844 51573 104
0.99693 62519 61680 15660 9813	0.00347 18948 93078 14325 500
0.99247 60552 11689 98109 8942	0.00544 71118 74217 21831 282
0.98604 55580 70398 65992 7101	0.00741 17693 63190 21036 211
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0.96732 82236 64986 43838 8618	0.01129 31846 49931 53764 963
0.95507 85091 14292 84264 0327	0.01320 21908 14676 74762 507
0.94093 25790 03815 35552 2816	0.01508 49878 65443 12768 230
0.92491 85168 97934 44027 2266	0.01693 78363 76302 93253 184
0.90706 81162 60922 84943 6353	0.01875 70570 93133 42341 545
0.88741 68168 63348 17112 4374	0.02053 90378 24326 45338 449
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0.84287 10819 98980 24231 8594	0.02397 72078 89100 29227 869
0.81806 50876 25441 18902 7232	0.02562 65709 08468 48279 899
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0.15533 01378 82070 24730 9006	0.04401 96023 90183 45875 736
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0.98782 89667 67524 52014 1267	0.00646 64649 07037 53840 196
0.98050 93245 97416 60577 1693	0.00817 07107 07327 82640 372
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0.94843 11650 79287 37792 1920	0.01318 69567 62824 80211 961
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0.91880 52912 28393 99127 1240	0.01642 18171 19024 64004 360
0.90159 29025 48446 33409 3964	0.01799 79931 25645 05063 795
0.88281 96914 47895 54216 0092	0.01954 30115 20127 88937 957
0.86251 81576 26883 42546 1262	0.02105 41975 12282 84223 645
0.84072 34466 54958 05009 7012	0.02252 89349 13865 77645 055
0.81747 32891 03541 31484 3395	0.02396 46706 53716 95917 477
0.79280 79352 13563 51036 0414	0.02535 89191 90216 37909 421
0.76677 00852 06641 92552 4095	0.02670 92668 10120 85177 235
0.73940 48153 58032 52649 6514	0.02801 33758 04780 54082 526
0.71075 94999 58041 12676 9559	0.02926 89885 15725 98680 503
0.68088 37292 96269 59804 4384	0.03047 39312 42214 53920 314
0.64982 92238 10262 23258 6887	0.03162 61180 03749 64805 603
0.61764 97445 46925 96445 0186	0.03272 35541 50934 22052 152
0.58440 10000 91577 88031 9607	0.03376 43398 18334 09264 696
0.55014 05501 25645 55150 3015	0.03474 66732 13330 40653 510
0.51492 77057 79916 64052 7992	0.03566 88537 35240 45308 912
0.47882 34269 55803 15701 5809	0.03652 92849 19290 33900 685
0.44189 02167 92348 26653 7336	0.03732 64772 00332 09016 731
0.40419 20134 61653 61319 0807	0.03805 90504 91513 60313 563
0.36579 40794 80035 92532 6018	0.03872 57365 73432 57584 147
0.32676 28887 26526 30235 8400	0.03932 53812 89635 16252 077
0.28716 60113 64297 24694 2702	0.03985 69465 44656 35257 597
0.24707 19968 64234 75690 4240	0.04031 95121 01141 57755 817
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0.98284 85727 38629 07041 8288	0.00719 29047 68117 31275 268
0.97490 91405 85727 79338 5645	0.00868 39452 69260 85842 641
0.96548 50890 43799 25145 2273	0.01016 17660 41103 06452 083
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0.94224 27613 09872 67475 2266	0.01306 87615 92401 33929 379
0.92845 98771 72445 79595 3046	0.01449 35080 40509 07611 696
0.91326 31025 71757 65416 4734	0.01589 61835 83725 68804 490
0.89667 55794 38770 68319 4324	0.01727 46520 56269 30635 858
0.87872 25676 78213 82870 3773	0.01862 68142 08299 03142 874
0.85943 14066 63111 09697 7192	0.01995 06108 78141 99892 889
0.83883 14735 80255 27561 6623	0.02124 40261 15782 00638 871
0.81695 41386 81463 47037 1125	0.02250 50902 46332 46192 622
0.79383 27175 04605 44994 8639	0.02373 18828 65930 10129 319
0.76950 24201 35041 37386 5616	0.02492 25357 64115 49110 512
0.74400 02975 83597 27231 6541	0.02607 52357 67565 11790 297
0.71736 51853 62099 88025 4068	0.02718 82275 00486 38067 442
0.68963 76443 42027 60077 1208	0.02825 98160 57276 86239 675
0.66085 98989 86119 80173 5967	0.02928 83695 83267 84769 277
0.63107 57730 46871 96624 7928	0.03027 23217 59557 98066 122
0.60033 06228 29751 74315 4746	0.03121 01741 88114 70164 244
0.56867 12681 22709 78472 5486	0.03210 04986 73487 77314 806
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0.43387 53708 31756 09306 2387	0.03516 05290 44747 59349 553
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0.25095 23583 92272 12049 3159	0.03777 63643 62001 39748 978
0.21299 45028 57666 13257 2389	0.03812 97113 14477 63834 421
0.17471 22918 32646 81255 9339	0.03842 49930 06959 42318 521
0.13616 40228 09143 88655 9241	0.03866 17597 74076 46332 708
0.09740 83984 41584 59906 3278	0.03883 96510 59051 96893 177
0.05850 44371 52420 66862 8993	0.03895 83959 62769 53119 863
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0.99488 23742 95616 27845 3367	0.00370 65001 25759 31670 687
0.99050 52177 16415 96378 7124	0.00504 68384 26924 44272 545
0.98479 09576 85580 49732 7164	0.00638 03985 87897 51509 869
0.97774 72884 12243 39193 3334	0.00770 53559 60382 75707 990
0.96938 37119 23678 32800 5136	0.00901 99154 39993 63127 897
0.95971 15159 57188 36239 6347	0.01032 23002 30524 24589 382
0.94874 37562 54578 95340 4649	0.01161 07512 86703 89800 962
0.93649 52381 16430 69968 6698	0.01288 35288 56498 08429 051
0.92298 24960 96090 04558 8769	0.01413 89145 48400 83293 056
0.90822 37715 39091 88489 6825	0.01537 52135 42389 62687 441
0.89223 89878 91355 90158 1714	0.01659 07568 31154 67007 521
0.87504 97237 69097 08763 8653	0.01778 39034 51398 17090 774
0.85667 91838 09956 18909 5811	0.01895 30426 88182 84044 681
0.83715 21673 37082 65019 8542	0.02009 65962 43575 42174 179
0.81649 50348 74791 61472 0573	0.02121 30203 64089 37967 242
0.79473 56725 59116 50544 7149	0.02230 08079 22839 37418 946
0.77190 34544 90293 07866 0311	0.02335 84904 52989 89189 770
0.74802 92030 77434 07395 6270	0.02438 46401 29435 68314 242
0.72314 51474 28601 02138 4583	0.02537 78716 95866 08847 737
0.69728 48798 42249 54765 0964	0.02633 68443 34514 35982 173
0.67048 33104 58662 66070 2672	0.02726 02634 76011 16478 577
0.64277 66201 32514 71484 3400	0.02814 68825 46865 07584 638
0.61420 22115 90136 99307 1297	0.02899 55046 52190 15208 987
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0.99543 18120 58344 66392 6755	0.00330 88672 43336 01819 543
0.99152 39288 11062 78612 9147	0.00450 61236 13674 97786 414
0.98642 13650 57832 84873 4254	0.00569 79815 60747 35260 085
0.98013 02513 45148 38545 8953	0.00688 29832 08463 28431 473
0.97265 81620 90193 13999 7465	0.00805 96949 44620 01565 867
0.96401 40981 71505 48339 3667	0.00922 66969 57741 99094 032
0.95420 84738 81500 33616 0720	0.01038 25823 09893 21461 381
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0.93116 11875 00432 00700 5847	0.01265 54458 37168 12886 888
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0.87173 96188 62903 43447 4028	0.01700 75862 85222 67570 940
0.85421 02590 67071 88228 6021	0.01804 76126 34460 23616 405
0.83565 16425 33377 04556 4199	0.01906 58930 39137 31842 532
0.81608 61309 29481 05644 3754	0.02006 12005 44639 59596 453
0.79553 72991 58248 13486 3601	0.02103 23358 78722 56311 706
0.77402 99069 50334 24680 6958	0.02197 81288 95934 13383 869
0.75158 98690 29638 46817 8415	0.02289 74399 87163 18463 499
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0.70402 11012 02391 14355 5469	0.02465 22188 35904 85293 597
0.67894 96879 46597 14618 1269	0.02548 55722 19443 22848 447
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0.51246 56800 93027 97098 8463	0.02980 83314 64031 27548 715
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5. References

[1] P. Davis and P. Rabinowitz, Abscissas and Weights for Gaussian Quadrature of High Order, J. Res. NBS **56**, 35–37 (1956) (RP 2645).

[2] Philip Davis and Paul Rabinowitz, Abscissas and Weights for Gaussian Quadrature of High Order: Values for $n = 64, 80, 96$, J. Res. NBS **60**, 613–614 (1958) (RP 2875).

[3] R. L. Brooke, C. A. Hoer, and C. H. Love, Inductance and Characteristic Impedance of Strip Lines, to be published.

[4] A. H. Stroud, Mathematics of Computation, Vol. 18, p. 164 (1964).

[5] H. J. Gawlik, Zeros of Legendre Polynomials or Orders 2–64 and Weight Coefficients of Gaussian Quadrature Formulae, A.R.D.E. Memorandum (B) 77/58. (Dec. 1958).

[6] F. B. Hildebrand, Introduction to Numerical Analysis (McGraw-Hill Book Co., Inc., New York, N.Y., 1956).

(Paper 70B4–186)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

Transverse Impact of a Linear Three-Element Spring and Dashpot Model Filament: Theory

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(July 7, 1966)

The mathematics of wave propagation in a viscoelastic filament subjected to constant velocity transverse impact is discussed. The equations governing the stress-strain-time behavior are assumed to be those for a linear model consisting of a spring coupled in parallel with a spring and dashpot in series. The nature of the solution is discussed, and a method is described for calculating the configuration, and the stress, strain, and particle velocity distributions along the impacted filament. The method used consists of an integration along the characteristics of the system of differential equations describing the problem.

Key Words: Characteristics, impact, linear viscoelasticity, partial differential equations, stress waves, wave propagation.

1. Introduction

When a flexible filament is struck transversely by a high velocity projectile, strain waves and waves of transverse motion are generated. The associated displacement of filament material can be recorded by high speed photography, and the resulting data can be analyzed to obtain the stress-strain, creep, and stress relaxation behavior of the material under impact conditions. Research of this type has been described by Petterson et al., [1, 2, 3][1] and Smith et al., [4, 5, 6].

A theory originated by Rachmatulin [7], Taylor [8, 9], and von Karman [10], and modified by Smith et al., [11, 12] has been used to analyze the experimental data. This theory, however, requires additional modification in order to account adequately for viscoelastic effects. Malvern [13, 14] has helped in this respect by proposing a simple rate-dependent constitutive equation for approximating these effects, and has used it in treating the longitudinal impact problem. Further insight may be gained through the work of Lee and Kanter [15], Glauz and Lee [16], Morrison [17], and Smith [18] who have investigated the behavior of various linear spring and dashpot models subjected to longitudinal impact.

Transverse impact of materials having rate-dependent constitutive equations has received less attention. Cristescu [19, 20, 21, 22, 23] has discussed various aspects of the problem of strain wave propagation and transverse wave motion in strings, including some strings with stress-strain-time behavior given by rate dependent constitutive equations. For some cases solutions calculated by a digital computer were obtained by integration along the characteristics of the system of differential equations describing the problem, but usually only continuous solutions were considered.

The problem of transverse impact at constant velocity involves a solution in which the strain is discontinuous at the strain wave front, and the slope of the filament is discontinuous at the front of a wave of transverse motion. This latter problem will be discussed here, and a solution obtained by an extension of the methods of Cristescu. The understanding gained is expected to be of use in the design and analysis of future experiments.

2. The Model

The constitutive equation governing the stress-strain-time behavior of the filament is that for the three-element linear spring and dashpot model depicted in figure 1. The longitudinal impact behavior of this model has been treated previously by Morrison [17] and Smith [18]. In the model the Hooke's law constants of the two springs are g_1 and g_2, and the viscosity constant of the dashpot γ_1. T is the total

[1] Figures in brackets indicate the literature references at the end of this paper.

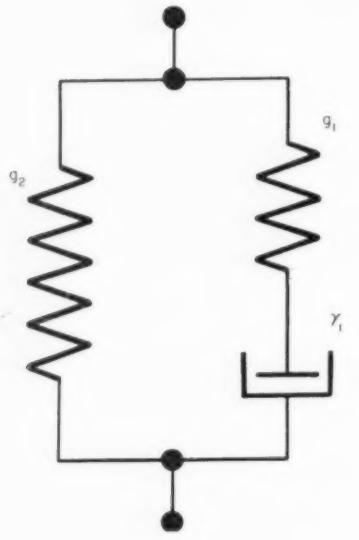

FIGURE 1. *Linear spring and dashpot model consisting of two springs and one dashpot.*

force, ϵ the strain, and t the time. The constitutive equation is

$$(g_1+g_2)\frac{\partial \epsilon}{\partial t}+\frac{g_1 g_2}{\gamma_1}\epsilon=\frac{\partial T}{\partial t}+\frac{g_1}{\gamma_1}T. \tag{1}$$

It is more convenient to represent the model in terms of other parameters. Thus let $g_1=\lambda g$, $g_2=(1-\lambda)g$, $\gamma_1=\lambda g\tau$. The parameter $\tau=\gamma_1/g_1$ is the relaxation time of the Maxwell element constituting one branch of the model, and λ is a dimensionless parameter that takes on values between 0 and 1. When $\lambda=0$ the model degenerates into a spring of modulus g, and when $\lambda=1$ the model becomes a Maxwell element. For practical applications a value of λ of approximately 0.1 coupled with a relaxation time τ less than the observation time following an impact is of interest. Such a model would be mostly springlike in its behavior, but would exhibit noticeable viscoelastic effects shortly after impact. Another convenient parameter is $c=\sqrt{g/m}$, the velocity at which a strain wave front propagates in the filament. The quantity m is the mass per unit length of the unstrained filament. Equation (1) expressed in the parameters τ, λ, c becomes

$$mc^2\frac{\partial \epsilon}{\partial t}+\frac{(1-\lambda)mc^2}{\tau}\epsilon=\frac{\partial T}{\partial t}+\frac{1}{\tau}T. \tag{2}$$

3. Formulation of the Problem

In the laboratory or observer's coordinate system, let x' and y' denote the horizontal and vertical coordinates respectively. At time $t=0$ let the filament at $x'=0$ be impacted at constant velocity V in the y' direction. The motion of the filament is to be described in terms of a Lagrangian coordinate system; i.e., a system of x coordinates fixed to the filament, moving and extending with it, but such that in the initial unstrained state the x and x' coordinates of the filament coincide. The displacement of any point x of the filament at time t relative to its original position is given by $\xi(x, t)$ for horizontal displacements, and $\eta(x, t)$ for vertical displacements. The location in the laboratory coordinate system of a point x on the filament is thus given by

$$x'=x+\xi(x, t), \qquad y'=\eta(x, t). \tag{3a, b}$$

Consider an element of unstrained length Δx. After impact, this element is strained, and its new length measured in the laboratory coordinate system becomes $\Delta x\sqrt{(1+\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2}$. The increase in length per unit length, or strain ϵ, is thus

$$\epsilon=\sqrt{(1+\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2}-1. \tag{4}$$

The angle θ that the element makes with the horizontal is given by

$$\cos\theta=\frac{1}{1+\epsilon}\left(1+\frac{\partial\xi}{\partial x}\right), \qquad \sin\theta=\frac{1}{1+\epsilon}\frac{\partial\eta}{\partial x} \tag{5a, b}$$

so that the differential equations that govern the motion of the small element are

$$m\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial x}(T\cos\theta)=\frac{\partial}{\partial x}\left[\frac{T}{1+\epsilon}\left(1+\frac{\partial\xi}{\partial x}\right)\right]$$

$$m\frac{\partial^2\eta}{\partial t^2}=\frac{\partial}{\partial x}(T\sin\theta)=\frac{\partial}{\partial x}\left[\frac{T}{1+\epsilon}\frac{\partial\eta}{\partial x}\right]. \tag{6a, b}$$

Equations (6a) and (6b) can be incorporated into a system of linear equations by introducing the horizontal and vertical components of the velocity (in x', y' space) of a point on the filament

$$u=\partial\xi/\partial t, \qquad v=\partial\eta/\partial t \tag{7a, b}$$

and the direction variables α, β defined by

$$\alpha=1+\frac{\partial\xi}{\partial x}, \qquad \beta=\frac{\partial\eta}{\partial x}. \tag{8a, b}$$

It is also helpful to introduce the flow or particle velocity w of a point along the direction of the filament. An increment of this velocity dw has horizontal and vertical components du and dv respectively and thus satisfies the relation

$$dw=du\cos\theta+dv\sin\theta. \tag{9}$$

The system of linear equations becomes

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\alpha T}{m(1+\epsilon)}\right] \tag{10a}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\beta T}{m(1+\epsilon)}\right] \tag{10b}$$

$$\frac{\partial u}{\partial x}=\frac{\partial \alpha}{\partial t} \tag{10c}$$

$$\frac{\partial v}{\partial x}=\frac{\partial \beta}{\partial t} \tag{10d}$$

$$d\epsilon=\frac{\alpha}{1+\epsilon}\,d\alpha+\frac{\beta}{1+\epsilon}\,d\beta \tag{10e}$$

$$dw=\frac{\alpha}{1+\epsilon}\,du+\frac{\beta}{1+\epsilon}\,dv \tag{10f}$$

$$mc^2\frac{\partial \epsilon}{\partial t}+\frac{(1-\lambda)mc^2}{\tau}\,\epsilon=\frac{\partial T}{\partial t}+\frac{1}{\tau}\,T. \tag{10g}$$

Equation (10e) is the differential form of eq (4).

The seven variables u, v, w, α, β, ϵ, and T must satisfy the initial conditions: For $t=0$, $x>0$

$$u=v=w=\beta=\epsilon=T=0,\ \alpha=1 \tag{11}$$

and the boundary conditions: For $t>0$, $x=0$

$$u=0,\ v=V \tag{12a, b}$$

and for $t>0$, $x\to\infty$

$$u=v=w=\beta=\epsilon=T=0,\ \alpha=1. \tag{13}$$

If conditions (12) are substituted into eq (10f), an additional boundary condition: For $t>0$, $x=0$

$$w=a\ constant \tag{14}$$

is obtained. The value of this constant velocity is attained at the instant of impact, before any creep or stress relaxation has occurred, and thus is the same as that given in the solution for the springlike filament discussed in the next section.

In addition to the initial and boundary conditions, certain compatibility conditions, to be given later, must be satisfied.

4. Solution for the Case $\lambda=0$

When $\lambda=0$ the model degenerates to a spring of Hooke's law constant g. The solution for this case, discussed in detail elsewhere [11] is illustrated in figure 2. The impact causes a wave of constant strain ϵ_0 and constant tension T_0 to propagate along the filament at velocity c, so that at time t the strain wave front has arrived at point F. In the wake of this wave front the strained material flows inward at velocity $-c\epsilon_0$. The increase in length of the filament is taken up by a wave of transverse motion, which at the time t has a front at point B.

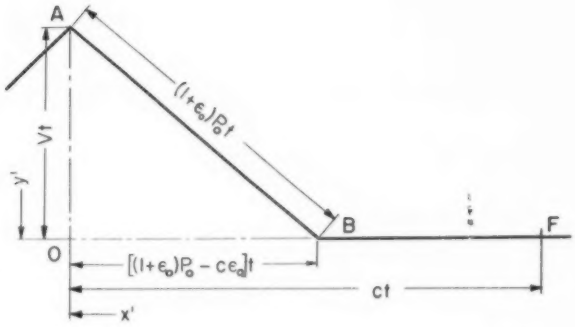

FIGURE 2. *Configuration of a spring-like filament at time t after transverse impact.*

The filament, originally horizontal, was struck with velocity V in the vertical direction. After time t the resulting strain wave propagating with velocity c has a front at F, and the filament becomes uniformly strained at strain ϵ_0. The strained material between points F and B moves inward horizontally with velocity $-c\epsilon_0$, but at the transverse wave front, point B, this inward motion ceases. Material in the transverse wave moves vertically with velocity V. The transverse wave front moves with velocity P_0 relative to points fixed on the filament, but with velocity $(1+\epsilon_0)P_0-c\epsilon_0$ with respect to the laboratory (x', y') coordinate system.

In the transverse wave the filament material moves in the y' direction at the impact velocity V, and there is no component of velocity in the x' direction. The profile of the wave is a straight line. The transverse wave front moves at velocity $P_0=\sqrt{T_0/m(1+\epsilon_0)}$ relative to the Lagragian coordinate system, so that the length of strained filament AB as measured in the x', y' system is equal to $(1+\epsilon_0)P_0T$, and the horizontal projection of this length is $[(1+\epsilon_0)P_0-c\epsilon_0]t$.

As OAB is a right triangle, the following relationship applies

$$V^2=(1+\epsilon_0)^2P_0^2-[(1+\epsilon_0)P_0-c\epsilon_0]^2. \tag{15}$$

If the mass per unit length m and Hooke's law constant g are known for the specimen, the velocities c and P_0 can be calculated for any strain ϵ_0, and upon substituting these values into eq (15), the value of ϵ_0 corresponding to any transverse impact velocity V can be found.

In the region between the strain wave front and the transverse wave front, the particle velocity w and the horizontal velocity u are both constant and equal to $-c\epsilon_0$. In the region of the transverse wave, $u=0$ but the velocity w remains equal to $-c\epsilon_0$, even though in the x', y' coordinate system there is no observable flow of material longitudinally along the filament. This paradox can be explained by the following interpretation of the particle velocity w. Suppose that

there is a second filament, inextensible and fixed to the impacted filament at a point a great distance in advance of the strain wave front. Before impact let the two filaments coincide with the inextensible filament ending at the point of impact, and after impact let the inextensible filament be constrained to move alongside the impacted filament. Under these conditions the point under impact will recede from the end of the inextensible filament with velocity w.

The solution for the limit case $\lambda = 0$ can be stated briefly as follows. The quantities w, ϵ, and T are continuous between the point of impact and the strain wave front at $x = ct$, and have the following values:

$$w = -c\epsilon_0,\ \epsilon = \epsilon_0,\ T = mc^2\epsilon_0 \qquad (16a,b,c)$$

where ϵ_0 is found by solving eq (15). The quantities u, v, α, β are discontinuous at the transverse wave front $x = P_0 t = t\sqrt{T_0/m(1+\epsilon_0)}$.
For $0 \leq x < P_0 t$

$$u = 0,\ v = V$$

$$\alpha = (1+\epsilon_0)\cos\theta = 1 + \epsilon_0 - c\epsilon_0/P_0 \qquad (17a,b,c,d)$$

$$\beta = (1+\epsilon_0)\sin\theta = -V/P_0$$

and for $P_0 t \leq x < ct$

$$u = w = -c\epsilon_0,\ v = \beta = 0,\ \alpha = 1 + \epsilon_0. \qquad (18a,b,c)$$

The limit solution just presented indicates the type of solution to be sought for the general case where $\lambda > 0$; namely, a solution in which w, ϵ, and T are continuous between the point of impact and the strain wave front, and u, v, α, and β are continuous only between the point of impact and the transverse wave front. A method for finding this solution by integrating along the characteristics of the system of differential equations will be developed in the succeeding sections.

5. The Characteristics of the System of Equations

If the variables u, v, α, and β are eliminated from eqs (10a) through (10g) the following set of equations results.[2]

$$\frac{\partial w}{\partial t} = \frac{1}{m}\frac{\partial T}{\partial x}$$

$$\frac{\partial w}{\partial x} = \frac{\partial \epsilon}{\partial t}$$

$$mc^2\frac{\partial \epsilon}{\partial t} + \frac{(1-\lambda)mc^2}{\tau}\epsilon = \frac{\partial T}{\partial t} + \frac{1}{\tau}T. \qquad (19a,b,c)$$

[2] Equations (19a) through (19c) plus the boundary conditions of eqs (11) through (14) involving w, ϵ, and T are the same as the equations and boundary conditions for longitudinal impact at constant velocity $-c\epsilon_0$. The distribution of w, ϵ, and T can be calculated from the solutions for this case given by Morrison [17] and Smith [18]. However, the subsequent determination of values for u, v, α, and β from eqs (10a) through (10d) would be difficult and will not be attempted here by this method.

These equations then can be arranged to obtain

$$mc\left(\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x}\right) - \left(\frac{\partial T}{\partial t} + c\frac{\partial T}{\partial x}\right) = \frac{1}{\tau}[T - (1-\lambda)mc^2\epsilon]$$

$$-mc\left(\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x}\right) - \left(\frac{\partial T}{\partial t} - c\frac{\partial T}{\partial x}\right) = \frac{1}{\tau}[T - (1-\lambda)mc^2\epsilon].$$

For differentiations along curves of the families given by

$$dx/dt = \pm c \qquad (20)$$

the expressions in parentheses may be replaced by total derivatives so that the equations reduce to: Along the curves given by $dx/dt = c$

$$mcdw - dT = [T - (1-\lambda)mc^2\epsilon]dt/\tau \qquad (21a)$$

and along the curves given by $dx/dt = -c$

$$mcdw + dT = -[T - (1-\lambda)mc^2\epsilon]dt/\tau. \qquad (21b)$$

The curves satisfying eq (20) are known as characteristics.

If $P = \sqrt{T/m(1+\epsilon)}$, eq (10a) can be expressed in the form

$$\frac{\partial u}{\partial t} = P^2\frac{\partial \alpha}{\partial x} + \alpha\frac{\partial}{\partial x}(P^2)$$

which can be combined with eq (10c) to obtain the expressions

$$\frac{\partial u}{\partial t} + P\frac{\partial u}{\partial x} = P\left(\frac{\partial \alpha}{\partial t} + P\frac{\partial \alpha}{\partial x}\right) + \alpha\frac{\partial}{\partial x}(P^2)$$

$$\frac{\partial u}{\partial t} - P\frac{\partial u}{\partial x} = -P\left(\frac{\partial \alpha}{\partial t} - P\frac{\partial \alpha}{\partial x}\right) + \alpha\frac{\partial}{\partial x}(P^2).$$

Along curves of the families given by

$$dx/dt = \pm P \qquad (22)$$

these expressions reduce to

$$\frac{du}{dt} = \pm P\frac{d\alpha}{dt} + \alpha\frac{\partial}{\partial x}(P^2). \qquad (23a)$$

Similarly eqs (10b) and (10d) yield

$$\frac{dv}{dt} = \pm P\frac{d\beta}{dt} + \beta\frac{\partial}{\partial x}(P^2). \qquad (23b)$$

Equations (23a) and (23b) can be combined to give: Along the curves given by $dx/dt = P$

$$\beta du - \alpha dv = P\beta d\alpha - P\alpha d\beta \qquad (24a)$$

and along the curves given by $dx/dt = -P$

$$\beta du - \alpha dv = -P\beta d\alpha + P\alpha d\beta \tag{24b}$$

The curves satisfying eq (22) are additional characteristics.

Other equations holding along these characteristics can be found by the following manipulation involving eqs (19a) and (19b).

$$\frac{\partial}{\partial x}(P^2) = \frac{(1+\epsilon)(\partial T/\partial x) - T(\partial \epsilon/\partial x)}{m(1+\epsilon)^2}$$

$$= \frac{1}{1+\epsilon}\left[\frac{1}{m}\frac{\partial T}{\partial x} - P^2\frac{\partial \epsilon}{\partial x}\right].$$

But along curves defined by $dx/dt = P$

$$\frac{1}{m}\frac{\partial T}{\partial x} = \frac{\partial w}{\partial t} = \frac{dw}{dt} - P\frac{\partial w}{\partial x}$$

and

$$P^2\frac{\partial \epsilon}{\partial x} = P\frac{d\epsilon}{dt} - P\frac{\partial \epsilon}{\partial t} = P\frac{d\epsilon}{dt} - P\frac{\partial w}{\partial x}$$

so that

$$\frac{\partial}{\partial x}(P^2) = \frac{1}{1+\epsilon}\left[\frac{dw}{dt} - P\frac{d\epsilon}{dt}\right].$$

Thus along curves defined by $dx/dt = P$, eqs (23a) and (23b) reduce to

$$\begin{aligned}(1+\epsilon)(du - Pd\alpha) &= \alpha(dw - Pd\epsilon)\\ (1+\epsilon)(dv - Pd\beta) &= \beta(dw - Pd\epsilon)\end{aligned} \tag{25a,b}$$

Similarly along curves defined by $dx/dt = -P$

$$\begin{aligned}(1+\epsilon)(du + Pd\alpha) &= \alpha(dw + Pd\epsilon)\\ (1+\epsilon)(dv + Pd\beta) &= \beta(dw + Pd\epsilon)\end{aligned} \tag{26a,b}$$

Along characteristic curves in x, t space the integral surface of the system of differential equations may have discontinuities. It should be noted that the characteristic curve $x = ct$ determines the discontinuous wave front for the stress, strain and particle velocity distributions, and the characteristic curve that passes through the origin and is a solution of the equation $dx/dt = P$, determines the front of the transverse wave where the quantities u, v, α, and β are discontinuous.

6. Compatibility Conditions

An obvious requirement of the solution is that the displacements ξ and η be continuous at all points along the filament. The partial derivatives of these quantities with respect to x and t, however, are not necessarily continuous at the strain wave front and at the transverse wave front. At these places of discontinuity, according to the principle of kinematic compatibility [24], the following relations hold:

$$\begin{aligned}\left[\frac{\partial \xi}{\partial x}\right]\frac{dx}{dt} + \left[\frac{\partial \xi}{\partial t}\right] &= 0\\ \left[\frac{\partial \eta}{\partial x}\right]\frac{dx}{dt} + \left[\frac{\partial \eta}{\partial t}\right] &= 0\end{aligned} \tag{27a,b}$$

where the bracketed quantities represent jumps in value occasioned by the passage of a wave front at the point under consideration. For instance, at the point x the expression $[\epsilon]$ is defined as

$$[\epsilon] = \lim_{\delta \to 0}\{\epsilon(x, t+\delta) - \epsilon(x, t)\}.$$

The strain wave front velocity is greater than the transverse wave front velocity. Therefore at the strain wave front, where $dx/dt = c$, it follows that $\eta = 0$ and, from eq (4), $\partial \xi/\partial x = \epsilon$. Equation (27a) then yields the relation

$$[\epsilon] = -\frac{1}{c}[w]. \tag{28}$$

Similarly at the transverse wave front, where $dx/dt = P$, eqs (27a) and (27b) yield

$$\begin{aligned}[\alpha - 1] &= -\frac{1}{P}[u]\\ [\beta] &= -\frac{1}{P}[v]\end{aligned} \tag{29a,b}$$

As $[\alpha - 1] = \alpha - 1 - \epsilon$, $[u] = u - w$, $[\beta] = \beta$, and $[v] = v$, where α, u, β, and v refer to values in the wake of the wave front, eqs (29a) and (29b) can be expressed more conveniently as

$$\begin{aligned}\alpha &= 1 + \epsilon - u/P + w/P\\ \beta &= -v/P\end{aligned} \tag{30a,b}$$

Another compatibility condition is obtained as follows. Consider a point x on the filament. Just before arrival of the strain wave front at time t, the tension T and particle velocity w are both zero. At a slightly later time $t + \Delta t = t + \Delta x/c$, the tension is $-\Delta x(\partial T/\partial x)$ and the particle velocity is $(\Delta x/c)(\partial w/\partial t)$. Thus the jump relations for T and w are $[T] = -\Delta x(\partial T/\partial x)$ and $[w] = (\Delta x/c)(\partial w/\partial t)$. Application of eq (19a) then yields the condition of dynamic compatibility

$$[T] = -mc[w]. \tag{31}$$

7. Values of the Tension, Strain, and Particle Velocity at the Strain Wave Front

Equation (21a) is a differential relation involving the tension T, strain ϵ, and particle velocity w, that holds at the strain wave front. Hence if values of T and w from eqs (28) and (31) are substituted into this relation, it becomes $2d\epsilon = -(\lambda\epsilon/\tau)dt$, which when integrated yields

$$\epsilon(ct, t) = \epsilon(0, 0)e^{-(\lambda t/2\tau)}. \tag{32}$$

Substituting this result into eqs (28) and (31) then gives

$$w(ct, t) = -c\epsilon(0,0)e^{-(\lambda t/2\tau)} \tag{33}$$

$$T(ct, t) = mc^2\epsilon(0,0)e^{-(\lambda t/2\tau)}. \tag{34}$$

When $x = t = 0$ the strain and transverse wave fronts coincide. If values at $t = x = 0$ are designated by a zero subscript, the boundary conditions give $u_0 = 0$, $v_0 = V$, and eqs (30a) and (30b) reduce to

$$\alpha_0 = 1 + \epsilon_0 - c\epsilon_0/P_0$$
$$\beta_0 = -V/P_0 \tag{35a,b}$$

Thus eq (4) expressed as $\alpha_0^2 + \beta_0^2 = (1+\epsilon_0)^2$, can be put in the form $V^2 = P_0^2(1+\epsilon_0)^2 - [(1+\epsilon_0)P_0 - c\epsilon_0]^2$. But $P_0 = c\sqrt{\epsilon_0/(1+\epsilon_0)}$ so that

$$V^2/c^2 = \epsilon_0(1+\epsilon_0) - [\sqrt{\epsilon_0(1+\epsilon_0)} - \epsilon_0]^2. \tag{36}$$

The value of ϵ_0 to be used in eqs (32), (33), and (34) is obtained by solving eq (36).

8. Integration Along the Characteristics

A numerical solution to the system of differential eqs (10a) through (10g) and the initial and boundary conditions (11) through (14) can be found by integrating along the characteristics of the system. The nature of the problem and the method of solution are easily understood from an inspection of figure 3, in which some of the characteristics are plotted.

The straight line characteristics of positive slope are solutions of the equation $dx/dt = c$. Their slope is equal to $1/c$. The other straight line characteristics completing the network are solutions of the equation $dx/dt = -c$. The characteristic passing through the origin, and having the equation $x = ct$ corresponds to the discontinuity marking the wave front of the strain, tension, and particle velocity distributions. Thus in the domain between this characteristic and the x axis

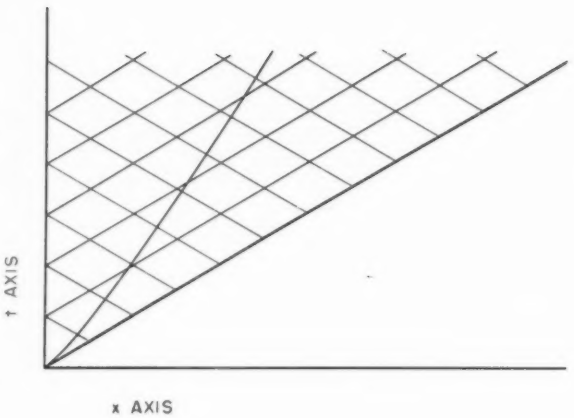

FIGURE 3. *A plot of the characteristics network.*
The straight line characteristics of positive slope are solutions of the equation $dx/dt = c$. The straight line characteristics of negative slope are solutions of the equation $dx/dt = -c$. The straight line characteristic through the origin has the equation $x = ct$, or equation of the strain wave front. The curved characteristic passing through the origin is that solution of the equation $dx/dt = \sqrt{T/m}(1+\epsilon)$ that gives the location of the transverse wave front.

may be thought of as boundary conditions holding along the characteristic $x = ct$.

The curved line passing through the origin represents a solution of the characteristic equation $dx/dt = P$. The exact course of this characteristic is not known in advance, as P is a function of ϵ and T. This characteristic corresponds to the front of the wave of transverse motion that propagates along the filament; thus in the domain between this characteristic and the characteristic $x = ct$ the velocity u is equivalent to the velocity w, the direction variable α is equal to $1+\epsilon$, and the velocity v and direction variable β are equal to zero. The jump relations (30a) and (30b) holding along the transverse wave front characteristic constitute a second set of boundary conditions.

A third set of boundary conditions expressed by eqs (12a), (12b), and (14) holds along the time axis.

The problem consists in finding the values of w, ϵ, and T at all of the mesh points in the domain between and along the time axis and the characteristic $x = ct$, in finding the course of the transverse wave front characteristic, and in finding the values of u, v, α, and β at all of the mesh points between and along the time axis and the transverse wave front characteristic. The values of these quantities must satisfy the differential relations (10e), and (10f), the relation (10g) holding along the line $x = a$ *constant*, the differential relations (21a), (21b), (24a), (24b), (25a), (25b), (26a), and (26b) holding along the characteristics, and the boundary conditions.

A first step towards the solution consists in solving eq (36) for the initial strain ϵ_0 at the point of impact. The values of ϵ, w, and T along the characteristic

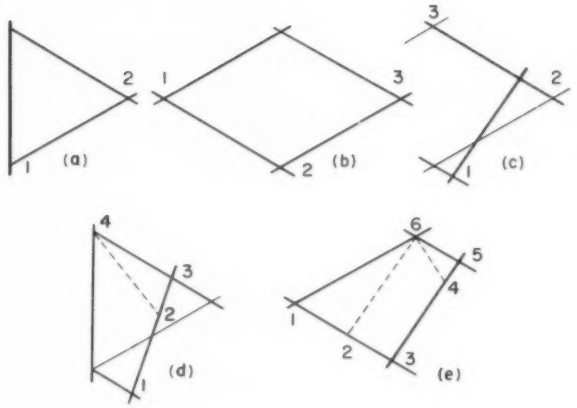

FIGURE 4. *Mesh situations encountered during integration of the system of differential equations along the characteristics network.*

depicts one of the triangular meshes lying along the t axis. In the lowest mesh points the values of ϵ, w, and T are known, and from the boundary condition (14) w is known to be constant along the t axis. The strain ϵ and tension T are found by solving eq (21b), which holds along the characteristic of negative slope, and eq (10g), which holds along the t axis. These equations can be expressed in the form

$$T - T_2 = -[\overline{T}_2 - (1-\lambda)mc^2E_2]\Delta t/\tau - mc(w_1 - w_2)$$

$$mc^2(\epsilon - \epsilon_1) - (T - T_1) = 2[\overline{T}_1 - (1-\lambda)mc^2E_1]\Delta t/\tau \qquad (37a, b)$$

where $\Delta t = t - t_2$ and $2\Delta t = t - t_1$. An iterative process is used in the solution. The quantities $\overline{T}_1$, $\overline{T}_2$, E_1, and E_2 are first set equal to T_1, T_2, ϵ_1, and ϵ_2, respectively, and approximate values are found for T and ϵ. The quantities $\overline{T}_1$, $\overline{T}_2$, E_1, and E_2 are then assigned the average values $(T + T_1)/2$, $(T + T_2)/2$, $(\epsilon + \epsilon_1)/2$, and $(\epsilon + \epsilon_2)/2$, respectively, and eqs (37a) and (37b) are solved again for more accurate values of T and ϵ. Another iteration may be performed if desired.

Figure 4b depicts a rhombus shaped mesh formed by the intersection of characteristics. If values of ϵ, w, and T are known at each of points 1, 2, and 3 the values of ϵ, w, and T at the vertex are found by solving the system of equations: eq (21a) which holds along the upper characteristic passing through point 1, eq (10g) which holds along the vertical line passing through point 2, and eq (21b) which holds along the upper characteristic passing through point 3 and 4b, values of ϵ, w, and T can be found at each of the mesh points in the domain between the t axis and the characteristic $x = ct$.

The values of α and β at the origin are next calculated from eqs (30a) and (30b). The values of u and v are already known from the boundary conditions. The solution for the mesh situation depicted in figure 4c can then be used to determine the course of the transverse wave front characteristic and the values of ϵ, w, T, u, v, α, and β at its intersections with the other characteristics. In figure 4c the quantities ϵ, w, T, u, v, α, and β are known at point 1, and the values of ϵ, w, and T are known at points 2 and 3. The values of ϵ, w, T, u, v, α, and β are required at the intersection of the transverse wave front characteristic with the characteristic passing through points 2 and 3. As a first step P is set equal to $\sqrt{T_1/m(1+\epsilon_1)}$ and the line $(x - x_1) = P(t - t_1)$ is drawn through point 1 to intersect the characteristic between points 2 and 3. The values of T and ϵ at the intersection are determined by interpolation. The quantity P is then set equal to $\sqrt{\overline{T}/m(1+E)}$, where $\overline{T}$ and E are the averages of T and T_1, and ϵ and ϵ_1 respectively. One or two iterations are performed to obtain more accurate values of T and ϵ and the coordinates (x, t) of the intersection. The value of w at the intersection is then found by interpolation. The values of u, v, α, and β at the intersection are found by solving the set of four equations (30a), (30b), (25a), and (25b) expressed in the form

$$u - u_1 - P(\alpha - \alpha_1) = \frac{A}{1+E}[w - w_1 - P(\epsilon - \epsilon_1)]$$

$$u + P\alpha = w + P(1+\epsilon)$$

$$v - v_1 - P(\beta - \beta_1) = \frac{B}{1+\epsilon}[w - w_1 - P(\epsilon - \epsilon_1)]$$

$$v + P\beta = 0. \qquad (38a,b,c,d)$$

In the solution process A and B are first set equal to α_1 and β_1 respectively and in subsequent iterations set equal to the average values along the transverse wave front characteristic.

Solutions are needed for two additional mesh situations in order to find the values of u, v, α, and β at the mesh points between the t axis and the transverse wave front characteristic. Figure 4d depicts a mesh of the type found along the t axis. The characteristic between points 1 and 3 may be either the transverse wave front characteristic or one of the solutions of the characteristic equation $dx/dt = v$. Usually the mesh is triangular in shape with point 1 lying on the t axis, but sometimes when the transverse wave front characteristic has a steep slope, the mesh has the more general form shown. The values of ϵ, w, T, u, v, α,

N 90

ABSCISSAS	WEIGHTS
0.99964 69712 86638 43746 3248	0.00090 59323 71214 83309 373
0.99814 03799 38568 15356 1306	0.00210 77787 74526 32989 148
0.99543 18120 58344 66392 6755	0.00330 88672 43336 01819 543
0.99152 39288 11062 78612 9147	0.00450 61236 13674 97786 414
0.98642 13650 57832 84873 4254	0.00569 79815 60747 35260 085
0.98013 02513 45148 38545 8953	0.00688 29832 08463 28431 473
0.97265 81620 90193 13999 7465	0.00805 96949 44620 01565 867
0.96401 40981 71505 48339 3667	0.00922 66969 57741 99094 032
0.95420 84738 81500 33616 0720	0.01038 25823 09893 21461 381
0.94325 31036 45357 76815 3575	0.01152 59578 89148 05885 059
0.93116 11875 00432 00700 5847	0.01265 54458 37168 12886 888
0.91794 72950 66586 38337 2356	0.01376 96851 12337 09343 075
0.90362 73479 31302 69386 9986	0.01486 73330 88043 32405 038
0.88821 86004 34745 98129 8376	0.01594 70671 51006 63901 321
0.87173 96188 62903 43447 4028	0.01700 75862 85222 67570 940
0.85421 02590 67071 88228 6021	0.01804 76126 34460 23616 405
0.83565 16425 33377 04556 4199	0.01906 58930 39137 31842 532
0.81608 61309 29481 05644 3754	0.02006 12005 44639 59596 453
0.79553 72991 58248 13486 3601	0.02103 23358 78722 56311 706
0.77402 99069 50334 24680 6958	0.02197 81288 95934 13383 869
0.75158 98690 29638 46817 8415	0.02289 74399 87163 18463 499
0.72824 42238 87390 36362 5800	0.02378 91614 52528 72321 010
0.70402 11012 02391 14355 5469	0.02465 22188 35904 85293 597
0.67894 96879 46597 14618 1269	0.02548 55722 19443 22848 447
0.65306 01932 16842 19196 1262	0.02628 82174 76514 58736 160
0.62638 38118 35045 12676 2609	0.02705 91874 81547 95852 161
0.59895 26867 60742 18588 8769	0.02779 75532 75302 27515 804
0.57079 98703 61220 97870 5362	0.02850 24251 84161 41631 876
0.54195 92845 85913 42618 9305	0.02917 29538 92100 74248 656
0.51246 56800 93027 97098 8463	0.02980 83314 64031 27548 715
0.48235 45943 77665 69252 0121	0.03040 77923 19286 95269 039
0.45166 23089 51869 36757 6379	0.03097 06141 54080 92094 594
0.42042 58056 28197 75610 1396	0.03149 61188 11818 63607 696
0.38868 27219 59498 20677 6509	0.03198 36731 00218 57603 946
0.35647 13058 88567 84613 1189	0.03243 26895 54255 61691 179
0.32383 03696 62345 96651 1395	0.03284 26271 44007 50457 863
0.29079 92430 66166 65154 9602	0.03321 29919 26551 31651 404
0.25741 77260 34420 12992 0888	0.03354 33376 41124 27668 293
0.22372 60406 94722 85926 8958	0.03383 32662 46831 68725 793
0.18976 47829 03379 01902 0874	0.03408 24284 02253 99546 361
0.15557 48733 30529 11951 1405	0.03429 05238 86375 04193 170
0.12119 75081 53924 08296 8749	0.03445 73019 60324 25617 460
0.08667 41094 20734 77008 5237	0.03458 25616 69496 89141 805
0.05204 62751 37206 94905 9279	0.03466 61520 85688 24018 827
0.01735 57291 46299 65246 1298	0.03470 79724 88950 05792 046

5. References

[1] P. Davis and P. Rabinowitz, Abscissas and Weights for Gaussian Quadrature of High Order, J. Res. NBS **56**, 35–37 (1956) (RP 2645).

[2] Philip Davis and Paul Rabinowitz, Abscissas and Weights for Gaussian Quadrature of High Order: Values for $n = 64, 80, 96$, J. Res. NBS **60**, 613–614 (1958) (RP 2875).

[3] R. L. Brooke, C. A. Hoer, and C. H. Love, Inductance and Characteristic Impedance of Strip Lines, to be published.

[4] A. H. Stroud, Mathematics of Computation, Vol. 18, p. 164 (1964).

[5] H. J. Gawlik, Zeros of Legendre Polynomials or Orders 2–64 and Weight Coefficients of Gaussian Quadrature Formulae, A.R.D.E. Memorandum (B) 77/58. (Dec. 1958).

[6] F. B. Hildebrand, Introduction to Numerical Analysis (McGraw-Hill Book Co., Inc., New York, N.Y., 1956).

(Paper 70B4–186)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

Transverse Impact of a Linear Three-Element Spring and Dashpot Model Filament: Theory

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(July 7, 1966)

The mathematics of wave propagation in a viscoelastic filament subjected to constant velocity transverse impact is discussed. The equations governing the stress-strain-time behavior are assumed to be those for a linear model consisting of a spring coupled in parallel with a spring and dashpot in series. The nature of the solution is discussed, and a method is described for calculating the configuration, and the stress, strain, and particle velocity distributions along the impacted filament. The method used consists of an integration along the characteristics of the system of differential equations describing the problem.

Key Words: Characteristics, impact, linear viscoelasticity, partial differential equations, stress waves, wave propagation.

1. Introduction

When a flexible filament is struck transversely by a high velocity projectile, strain waves and waves of transverse motion are generated. The associated displacement of filament material can be recorded by high speed photography, and the resulting data can be analyzed to obtain the stress-strain, creep, and stress relaxation behavior of the material under impact conditions. Research of this type has been described by Petterson et al., [1, 2, 3][1] and Smith et al., [4, 5, 6].

A theory originated by Rachmatulin [7], Taylor [8, 9], and von Karman [10], and modified by Smith et al., [11, 12] has been used to analyze the experimental data. This theory, however, requires additional modification in order to account adequately for viscoelastic effects. Malvern [13, 14] has helped in this respect by proposing a simple rate-dependent constitutive equation for approximating these effects, and has used it in treating the longitudinal impact problem. Further insight may be gained through the work of Lee and Kanter [15], Glauz and Lee [16], Morrison [17], and Smith [18] who have investigated the behavior of various linear spring and dashpot models subjected to longitudinal impact.

Transverse impact of materials having rate-dependent constitutive equations has received less attention. Cristescu [19, 20, 21, 22, 23] has discussed various aspects of the problem of strain wave propagation and transverse wave motion in strings, including some strings with stress-strain-time behavior given by rate dependent constitutive equations. For some cases solutions calculated by a digital computer were obtained by integration along the characteristics of the system of differential equations describing the problem, but usually only continuous solutions were considered.

The problem of transverse impact at constant velocity involves a solution in which the strain is discontinuous at the strain wave front, and the slope of the filament is discontinuous at the front of a wave of transverse motion. This latter problem will be discussed here, and a solution obtained by an extension of the methods of Cristescu. The understanding gained is expected to be of use in the design and analysis of future experiments.

2. The Model

The constitutive equation governing the stress-strain-time behavior of the filament is that for the three-element linear spring and dashpot model depicted in figure 1. The longitudinal impact behavior of this model has been treated previously by Morrison [17] and Smith [18]. In the model the Hooke's law constants of the two springs are g_1 and g_2, and the viscosity constant of the dashpot γ_1. T is the total

[1] Figures in brackets indicate the literature references at the end of this paper.

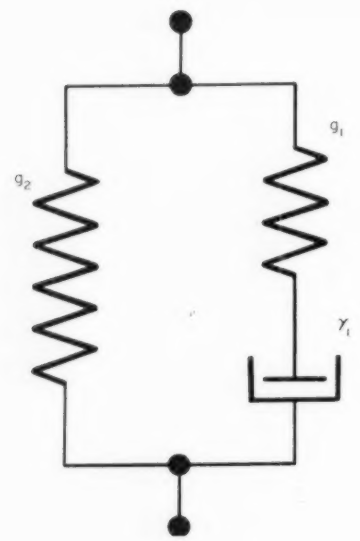

FIGURE 1. *Linear spring and dashpot model consisting of two springs and one dashpot.*

force, ϵ the strain, and t the time. The constitutive equation is

$$(g_1+g_2)\frac{\partial \epsilon}{\partial t}+\frac{g_1 g_2}{\gamma_1}\epsilon=\frac{\partial T}{\partial t}+\frac{g_1}{\gamma_1}T. \tag{1}$$

It is more convenient to represent the model in terms of other parameters. Thus let $g_1=\lambda g$, $g_2=(1-\lambda)g$, $\gamma_1=\lambda g\tau$. The parameter $\tau=\gamma_1/g_1$ is the relaxation time of the Maxwell element constituting one branch of the model, and λ is a dimensionless parameter that takes on values between 0 and 1. When $\lambda=0$ the model degenerates into a spring of modulus g, and when $\lambda=1$ the model becomes a Maxwell element. For practical applications a value of λ of approximately 0.1 coupled with a relaxation time τ less than the observation time following an impact is of interest. Such a model would be mostly springlike in its behavior, but would exhibit noticeable viscoelastic effects shortly after impact. Another convenient parameter is $c=\sqrt{g/m}$, the velocity at which a strain wave front propagates in the filament. The quantity m is the mass per unit length of the unstrained filament. Equation (1) expressed in the parameters τ, λ, c becomes

$$mc^2\frac{\partial \epsilon}{\partial t}+\frac{(1-\lambda)mc^2}{\tau}\epsilon=\frac{\partial T}{\partial t}+\frac{1}{\tau}T. \tag{2}$$

3. Formulation of the Problem

In the laboratory or observer's coordinate system, let x' and y' denote the horizontal and vertical coordinates respectively. At time $t=0$ let the filament at $x'=0$ be impacted at constant velocity V in the y' direction. The motion of the filament is to be described in terms of a Lagrangian coordinate system: i.e., a system of x coordinates fixed to the filament, moving and extending with it, but such that in the initial unstrained state the x and x' coordinates of the filament coincide. The displacement of any point x of the filament at time t relative to its original position is given by $\xi(x, t)$ for horizontal displacements, and $\eta(x, t)$ for vertical displacements. The location in the laboratory coordinate system of a point x on the filament is thus given by

$$x'=x+\xi(x, t), \qquad y'=\eta(x, t). \tag{3a, b}$$

Consider an element of unstrained length Δx. After impact, this element is strained, and its new length measured in the laboratory coordinate system becomes $\Delta x\sqrt{(1+\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2}$. The increase in length per unit length, or strain ϵ, is thus

$$\epsilon=\sqrt{(1+\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2}-1. \tag{4}$$

The angle θ that the element makes with the horizontal is given by

$$\cos\theta=\frac{1}{1+\epsilon}\left(1+\frac{\partial\xi}{\partial x}\right), \qquad \sin\theta=\frac{1}{1+\epsilon}\frac{\partial\eta}{\partial x} \tag{5a, b}$$

so that the differential equations that govern the motion of the small element are

$$m\frac{\partial^2\xi}{\partial t^2}=\frac{\partial}{\partial x}(T\cos\theta)=\frac{\partial}{\partial x}\left[\frac{T}{1+\epsilon}\left(1+\frac{\partial\xi}{\partial x}\right)\right]$$

$$m\frac{\partial^2\eta}{\partial t^2}=\frac{\partial}{\partial x}(T\sin\theta)=\frac{\partial}{\partial x}\left[\frac{T}{1+\epsilon}\frac{\partial\eta}{\partial x}\right]. \tag{6a, b}$$

Equations (6a) and (6b) can be incorporated into a system of linear equations by introducing the horizontal and vertical components of the velocity (in x', y' space) of a point on the filament

$$u=\partial\xi/\partial t, \qquad v=\partial\eta/\partial t \tag{7a, b}$$

and the direction variables α, β defined by

$$\alpha=1+\frac{\partial\xi}{\partial x}, \qquad \beta=\frac{\partial\eta}{\partial x}. \tag{8a, b}$$

It is also helpful to introduce the flow or particle velocity w of a point along the direction of the filament. An increment of this velocity dw has horizontal and vertical components du and dv respectively and thus satisfies the relation

$$dw=du\cos\theta+dv\sin\theta. \tag{9}$$

The system of linear equations becomes

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\alpha T}{m(1+\epsilon)}\right] \tag{10a}$$

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left[\frac{\beta T}{m(1+\epsilon)}\right] \tag{10b}$$

$$\frac{\partial u}{\partial x}=\frac{\partial \alpha}{\partial t} \tag{10c}$$

$$\frac{\partial v}{\partial x}=\frac{\partial \beta}{\partial t} \tag{10d}$$

$$d\epsilon=\frac{\alpha}{1+\epsilon}\,d\alpha+\frac{\beta}{1+\epsilon}\,d\beta \tag{10e}$$

$$dw=\frac{\alpha}{1+\epsilon}\,du+\frac{\beta}{1+\epsilon}\,dv \tag{10f}$$

$$mc^2\frac{\partial \epsilon}{\partial t}+\frac{(1-\lambda)mc^2}{\tau}\epsilon=\frac{\partial T}{\partial t}+\frac{1}{\tau}T. \tag{10g}$$

Equation (10e) is the differential form of eq (4).

The seven variables u, v, w, α, β, ϵ, and T must satisfy the initial conditions: For $t=0$, $x>0$

$$u=v=w=\beta=\epsilon=T=0,\ \alpha=1 \tag{11}$$

and the boundary conditions: For $t>0$, $x=0$

$$u=0,\ v=V \tag{12a, b}$$

and for $t>0$, $x\to\infty$

$$u=v=w=\beta=\epsilon=T=0,\ \alpha=1. \tag{13}$$

If conditions (12) are substituted into eq (10f), an additional boundary condition: For $t>0$, $x=0$

$$w=a\ constant \tag{14}$$

is obtained. The value of this constant velocity is attained at the instant of impact, before any creep or stress relaxation has occurred, and thus is the same as that given in the solution for the springlike filament discussed in the next section.

In addition to the initial and boundary conditions, certain compatibility conditions, to be given later, must be satisfied.

4. Solution for the Case $\lambda=0$

When $\lambda=0$ the model degenerates to a spring of Hooke's law constant g. The solution for this case, discussed in detail elsewhere [11] is illustrated in figure 2. The impact causes a wave of constant strain ϵ_0 and constant tension T_0 to propagate along the filament at velocity c, so that at time t the strain wave front has arrived at point F. In the wake of this wave front the strained material flows inward at velocity $-c\epsilon_0$. The increase in length of the filament is taken up by a wave of transverse motion, which at the time t has a front at point B.

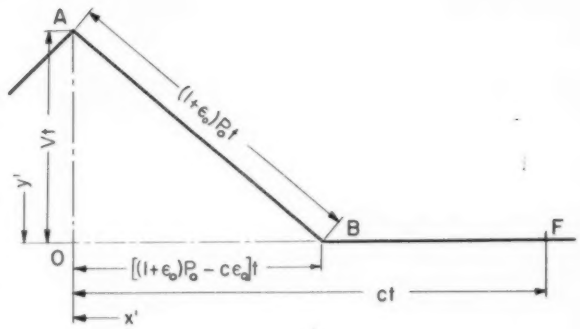

FIGURE 2. *Configuration of a spring-like filament at time t after transverse impact.*

The filament, originally horizontal, was struck with velocity V in the vertical direction. After time t the resulting strain wave propagating with velocity c has a front at F, and the filament becomes uniformly strained at strain ϵ_0. The strained material between points F and B moves inward horizontally with velocity $-c\epsilon_0$, but at the transverse wave front, point B, this inward motion ceases. Material in the transverse wave moves vertically with velocity V. The transverse wave front moves with velocity P_0 relative to points fixed on the filament, but with velocity $(1+\epsilon_0)P_0-c\epsilon_0$ with respect to the laboratory (x', y') coordinate system.

In the transverse wave the filament material moves in the y' direction at the impact velocity V, and there is no component of velocity in the x' direction. The profile of the wave is a straight line. The transverse wave front moves at velocity $P_0=\sqrt{T_0/m(1+\epsilon_0)}$ relative to the Lagragian coordinate system, so that the length of strained filament AB as measured in the x', y' system is equal to $(1+\epsilon_0)P_0T$, and the horizontal projection of this length is $[(1+\epsilon_0)P_0-c\epsilon_0]t$.

As OAB is a right triangle, the following relationship applies

$$V^2=(1+\epsilon_0)^2l_0^2-[(1+\epsilon_0)P_0-c\epsilon_0]^2. \tag{15}$$

If the mass per unit length m and Hooke's law constant g are known for the specimen, the velocities c and P_0 can be calculated for any strain ϵ_0, and upon substituting these values into eq (15), the value of ϵ_0 corresponding to any transverse impact velocity V can be found.

In the region between the strain wave front and the transverse wave front, the particle velocity w and the horizontal velocity u are both constant and equal to $-c\epsilon_0$. In the region of the transverse wave, $u=0$ but the velocity w remains equal to $-c\epsilon_0$, even though in the x', y' coordinate system there is no observable flow of material longitudinally along the filament. This paradox can be explained by the following interpretation of the particle velocity w. Suppose that

there is a second filament, inextensible and fixed to the impacted filament at a point a great distance in advance of the strain wave front. Before impact let the two filaments coincide with the inextensible filament ending at the point of impact, and after impact let the inextensible filament be constrained to move alongside the impacted filament. Under these conditions the point under impact will recede from the end of the inextensible filament with velocity w.

The solution for the limit case $\lambda = 0$ can be stated briefly as follows. The quantities w, ϵ, and T are continuous between the point of impact and the strain wave front at $x = ct$, and have the following values:

$$w = -c\epsilon_0,\ \epsilon = \epsilon_0,\ T = mc^2\epsilon_0 \qquad (16a,b,c)$$

where ϵ_0 is found by solving eq (15). The quantities u, v, α, β are discontinuous at the transverse wave front $x = P_0 t = t\sqrt{T_0/m(1+\epsilon_0)}$.
For $0 \leq x < P_0 t$

$$u = 0,\ v = V$$

$$\alpha = (1+\epsilon_0)\cos\theta = 1 + \epsilon_0 - c\epsilon_0/P_0 \qquad (17a,b,c,d)$$

$$\beta = (1+\epsilon_0)\sin\theta = -V/P_0$$

and for $P_0 t \leq x < ct$

$$u = w = -c\epsilon_0,\ v = \beta = 0,\ \alpha = 1 + \epsilon_0. \qquad (18a,b,c)$$

The limit solution just presented indicates the type of solution to be sought for the general case where $\lambda > 0$; namely, a solution in which w, ϵ, and T are continuous between the point of impact and the strain wave front, and u, v, α, and β are continuous only between the point of impact and the transverse wave front. A method for finding this solution by integrating along the characteristics of the system of differential equations will be developed in the succeeding sections.

5. The Characteristics of the System of Equations

If the variables u, v, α, and β are eliminated from eqs (10a) through (10g) the following set of equations results.[2]

$$\frac{\partial w}{\partial t} = \frac{1}{m}\frac{\partial T}{\partial x}$$

$$\frac{\partial w}{\partial x} = \frac{\partial \epsilon}{\partial t}$$

$$mc^2\frac{\partial \epsilon}{\partial t} + \frac{(1-\lambda)mc^2}{\tau}\epsilon = \frac{\partial T}{\partial t} + \frac{1}{\tau}T. \qquad (19a,b,c)$$

[2] Equations (19a) through (19c) plus the boundary conditions of eqs (11) through (14) involving w, ϵ, and T are the same as the equations and boundary conditions for longitudinal impact at constant velocity $-c\epsilon_0$. The distribution of w, ϵ, and T can be calculated from the solutions for this case given by Morrison [17] and Smith [18]. However, the subsequent determination of values for u, v, α, and β from eqs (10a) through (10d) would be difficult and will not be attempted here by this method.

These equations then can be arranged to obtain

$$mc\left(\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x}\right) - \left(\frac{\partial T}{\partial t} + c\frac{\partial T}{\partial x}\right) = \frac{1}{\tau}[T - (1-\lambda)mc^2\epsilon]$$

$$-mc\left(\frac{\partial w}{\partial t} - c\frac{\partial w}{\partial x}\right) - \left(\frac{\partial T}{\partial t} - c\frac{\partial T}{\partial x}\right) = \frac{1}{\tau}[T - (1-\lambda)mc^2\epsilon].$$

For differentiations along curves of the families given by

$$dx/dt = \pm c \qquad (20)$$

the expressions in parentheses may be replaced by total derivatives so that the equations reduce to: Along the curves given by $dx/dt = c$

$$mcdw - dT = [T - (1-\lambda)mc^2\epsilon]dt/\tau \qquad (21a)$$

and along the curves given by $dx/dt = -c$

$$mcdw + dT = -[T - (1-\lambda)mc^2\epsilon]dt/\tau. \qquad (21b)$$

The curves satisfying eq (20) are known as characteristics.

If $P = \sqrt{T/m(1+\epsilon)}$, eq (10a) can be expressed in the form

$$\frac{\partial u}{\partial t} = P^2\frac{\partial \alpha}{\partial x} + \alpha\frac{\partial}{\partial x}(P^2)$$

which can be combined with eq (10c) to obtain the expressions

$$\frac{\partial u}{\partial t} + P\frac{\partial u}{\partial x} = P\left(\frac{\partial \alpha}{\partial t} + P\frac{\partial \alpha}{\partial x}\right) + \alpha\frac{\partial}{\partial x}(P^2)$$

$$\frac{\partial u}{\partial t} - P\frac{\partial u}{\partial x} = -P\left(\frac{\partial \alpha}{\partial t} - P\frac{\partial \alpha}{\partial x}\right) + \alpha\frac{\partial}{\partial x}(P^2).$$

Along curves of the families given by

$$dx/dt = \pm P \qquad (22)$$

these expressions reduce to

$$\frac{du}{dt} = \pm P\frac{d\alpha}{dt} + \alpha\frac{\partial}{\partial x}(P^2). \qquad (23a)$$

Similarly eqs (10b) and (10d) yield

$$\frac{dv}{dt} = \pm P\frac{d\beta}{dt} + \beta\frac{\partial}{\partial x}(P^2). \qquad (23b)$$

Equations (23a) and (23b) can be combined to give: Along the curves given by $dx/dt = P$

$$\beta du - \alpha dv = P\beta d\alpha - P\alpha d\beta \qquad (24a)$$

and along the curves given by $dx/dt = -P$

$$\beta du - \alpha dv = -P\beta d\alpha + P\alpha d\beta \tag{24b}$$

The curves satisfying eq (22) are additional characteristics.

Other equations holding along these characteristics can be found by the following manipulation involving eqs (19a) and (19b).

$$\frac{\partial}{\partial x}(P^2) = \frac{(1+\epsilon)(\partial T/\partial x) - T(\partial \epsilon/\partial x)}{m(1+\epsilon)^2}$$

$$= \frac{1}{1+\epsilon}\left[\frac{1}{m}\frac{\partial T}{\partial x} - P^2\frac{\partial \epsilon}{\partial x}\right].$$

But along curves defined by $dx/dt = P$

$$\frac{1}{m}\frac{\partial T}{\partial x} = \frac{\partial w}{\partial t} = \frac{dw}{dt} - P\frac{\partial w}{\partial x}$$

and

$$P^2\frac{\partial \epsilon}{\partial x} = P\frac{d\epsilon}{dt} - P\frac{\partial \epsilon}{\partial t} = P\frac{d\epsilon}{dt} - P\frac{\partial w}{\partial x}$$

so that

$$\frac{\partial}{\partial x}(P^2) = \frac{1}{1+\epsilon}\left[\frac{dw}{dt} - P\frac{d\epsilon}{dt}\right].$$

Thus along curves defined by $dx/dt = P$, eqs (23a) and (23b) reduce to

$$\begin{aligned}(1+\epsilon)(du - Pd\alpha) &= \alpha(dw - Pd\epsilon)\\ (1+\epsilon)(dv - Pd\beta) &= \beta(dw - Pd\epsilon)\end{aligned} \tag{25a,b}$$

Similarly along curves defined by $dx/dt = -P$

$$\begin{aligned}(1+\epsilon)(du + Pd\alpha) &= \alpha(dw + Pd\epsilon)\\ (1+\epsilon)(dv + Pd\beta) &= \beta(dw + Pd\epsilon)\end{aligned} \tag{26a,b}$$

Along characteristic curves in x, t space the integral surface of the system of differential equations may have discontinuities. It should be noted that the characteristic curve $x = ct$ determines the discontinuous wave front for the stress, strain and particle velocity distributions, and the characteristic curve that passes through the origin and is a solution of the equation $dx/dt = P$, determines the front of the transverse wave where the quantities u, v, α, and β are discontinuous.

6. Compatibility Conditions

An obvious requirement of the solution is that the displacements ξ and η be continuous at all points along the filament. The partial derivatives of these quantities with respect to x and t, however, are not necessarily continuous at the strain wave front and at the transverse wave front. At these places of discontinuity, according to the principle of kinematic compatibility [24], the following relations hold:

$$\begin{aligned}\left[\frac{\partial \xi}{\partial x}\right]\frac{dx}{dt} + \left[\frac{\partial \xi}{\partial t}\right] &= 0\\ \left[\frac{\partial \eta}{\partial x}\right]\frac{dx}{dt} + \left[\frac{\partial \eta}{\partial t}\right] &= 0\end{aligned} \tag{27a,b}$$

where the bracketed quantities represent jumps in value occasioned by the passage of a wave front at the point under consideration. For instance, at the point x the expression $[\epsilon]$ is defined as

$$[\epsilon] = \lim_{\delta \to 0}\{\epsilon(x, t+\delta) - \epsilon(x, t)\}.$$

The strain wave front velocity is greater than the transverse wave front velocity. Therefore at the strain wave front, where $dx/dt = c$, it follows that $\eta = 0$ and, from eq (4), $\partial \xi/\partial x = \epsilon$. Equation (27a) then yields the relation

$$[\epsilon] = -\frac{1}{c}[w]. \tag{28}$$

Similarly at the transverse wave front, where $dx/dt = P$, eqs (27a) and (27b) yield

$$\begin{aligned}[\alpha - 1] &= -\frac{1}{P}[u]\\ [\beta] &= -\frac{1}{P}[v]\end{aligned} \tag{29a,b}$$

As $[\alpha - 1] = \alpha - 1 - \epsilon$, $[u] = u - w$, $[\beta] = \beta$, and $[v] = v$, where α, u, β, and v refer to values in the wake of the wave front, eqs (29a) and (29b) can be expressed more conveniently as

$$\begin{aligned}\alpha &= 1 + \epsilon - u/P + w/P\\ \beta &= -v/P\end{aligned} \tag{30a,b}$$

Another compatibility condition is obtained as follows. Consider a point x on the filament. Just before arrival of the strain wave front at time t, the tension T and particle velocity w are both zero. At a slightly later time $t + \Delta t = t + \Delta x/c$, the tension is $-\Delta x(\partial T/\partial x)$ and the particle velocity is $(\Delta x/c)(\partial w/\partial t)$. Thus the jump relations for T and w are $[T] = -\Delta x(\partial T/\partial x)$ and $[w] = (\Delta x/c)(\partial w/\partial t)$. Application of eq (19a) then yields the condition of dynamic compatibility

$$[T] = -mc[w]. \tag{31}$$

7. Values of the Tension, Strain, and Particle Velocity at the Strain Wave Front

Equation (21a) is a differential relation involving the tension T, strain ϵ, and particle velocity w, that holds at the strain wave front. Hence if values of T and w from eqs (28) and (31) are substituted into this relation, it becomes $2d\epsilon = -(\lambda\epsilon/\tau)dt$, which when integrated yields

$$\epsilon(ct, t) = \epsilon(0, 0)e^{-(\lambda t/2\tau)}. \tag{32}$$

Substituting this result into eqs (28) and (31) then gives

$$w(ct, t) = -c\epsilon(0,0)e^{-(\lambda t/2\tau)} \tag{33}$$

$$T(ct, t) = mc^2\epsilon(0,0)e^{-(\lambda t/2\tau)}. \tag{34}$$

When $x = t = 0$ the strain and transverse wave fronts coincide. If values at $t = x = 0$ are designated by a zero subscript, the boundary conditions give $u_0 = 0$, $v_0 = V$, and eqs (30a) and (30b) reduce to

$$\alpha_0 = 1 + \epsilon_0 - c\epsilon_0/P_0$$
$$\beta_0 = -V/P_0 \tag{35a,b}$$

Thus eq (4) expressed as $\alpha_0^2 + \beta_0^2 = (1+\epsilon_0)^2$, can be put in the form $V^2 = P_0^2(1+\epsilon_0)^2 - [(1+\epsilon_0)P_0 - c\epsilon_0]^2$. But $P_0 = c\sqrt{\epsilon_0/(1+\epsilon_0)}$ so that

$$V^2/c^2 = \epsilon_0(1+\epsilon_0) - [\sqrt{\epsilon_0(1+\epsilon_0)} - \epsilon_0]^2. \tag{36}$$

The value of ϵ_0 to be used in eqs (32), (33), and (34) is obtained by solving eq (36).

8. Integration Along the Characteristics

A numerical solution to the system of differential eqs (10a) through (10g) and the initial and boundary conditions (11) through (14) can be found by integrating along the characteristics of the system. The nature of the problem and the method of solution are easily understood from an inspection of figure 3, in which some of the characteristics are plotted.

The straight line characteristics of positive slope are solutions of the equation $dx/dt = c$. Their slope is equal to $1/c$. The other straight line characteristics completing the network are solutions of the equation $dx/dt = -c$. The characteristic passing through the origin, and having the equation $x = ct$ corresponds to the discontinuity marking the wave front of the strain, tension, and particle velocity distributions. Thus in the domain between this characteristic and the x axis the strain ϵ, tension T, and particle velocity w are zero. Along the characteristic the values of ϵ, w, and T are given by eqs (32), (33), and (34). These equations thus

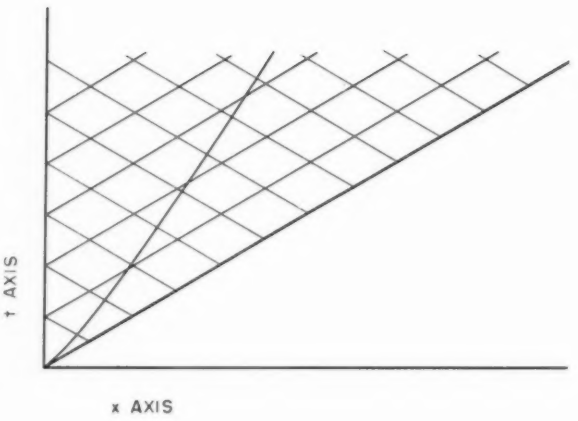

FIGURE 3. *A plot of the characteristics network.*
The straight line characteristics of positive slope are solutions of the equation $dx/dt = c$. The straight line characteristics of negative slope are solutions of the equation $dx/dt = -c$. The straight line characteristic through the origin has the equation $x = ct$, or equation of the strain wave front. The curved characteristic passing through the origin is that solution of the equation $dx/dt = \sqrt{T/m}(1+\epsilon)$ that gives the location of the transverse wave front.

may be thought of as boundary conditions holding along the characteristic $x = ct$.

The curved line passing through the origin represents a solution of the characteristic equation $dx/dt = P$. The exact course of this characteristic is not known in advance, as P is a function of ϵ and T. This characteristic corresponds to the front of the wave of transverse motion that propagates along the filament; thus in the domain between this characteristic and the characteristic $x = ct$ the velocity u is equivalent to the velocity w, the direction variable α is equal to $1+\epsilon$, and the velocity v and direction variable β are equal to zero. The jump relations (30a) and (30b) holding along the transverse wave front characteristic constitute a second set of boundary conditions.

A third set of boundary conditions expressed by eqs (12a), (12b), and (14) holds along the time axis.

The problem consists in finding the values of w, ϵ, and T at all of the mesh points in the domain between and along the time axis and the characteristic $x = ct$, in finding the course of the transverse wave front characteristic, and in finding the values of u, v, α, and β at all of the mesh points between and along the time axis and the transverse wave front characteristic. The values of these quantities must satisfy the differential relations (10e), and (10f), the relation (10g) holding along the line $x = a\ constant$, the differential relations (21a), (21b), (24a), (24b), (25a), (25b), (26a), and (26b) holding along the characteristics, and the boundary conditions.

A first step towards the solution consists in solving eq (36) for the initial strain ϵ_0 at the point of impact. The values of ϵ, w, and T along the characteristic $x = ct$ are then given by eqs (32), (33), and (34).

Values of ϵ, w, and T are then found at the first mesh point on the time axis above the origin. Figure 4a

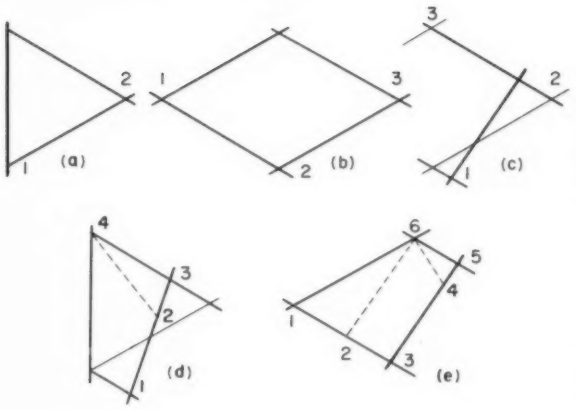

FIGURE 4. *Mesh situations encountered during integration of the system of differential equations along the characteristics network.*

depicts one of the triangular meshes lying along the t axis. In the lowest mesh points the values of ϵ, w, and T are known, and from the boundary condition (14) w is known to be constant along the t axis. The strain ϵ and tension T are found by solving eq (21b), which holds along the characteristic of negative slope, and eq (10g), which holds along the t axis. These equations can be expressed in the form

$$T-T_2=-[\overline{T}_2-(1-\lambda)mc^2E_2]\Delta t/\tau-mc(w_1-w_2)$$

$$mc^2(\epsilon-\epsilon_1)-(T-T_1)=2[\overline{T}_1-(1-\lambda)mc^2E_1]\Delta t/\tau \qquad (37a, b)$$

where $\Delta t=t-t_2$ and $2\Delta t=t-t_1$. An iterative process is used in the solution. The quantities $\overline{T}_1$, $\overline{T}_2$, E_1, and E_2 are first set equal to T_1, T_2, ϵ_1, and ϵ_2, respectively, and approximate values are found for T and ϵ. The quantities $\overline{T}_1$, $\overline{T}_2$, E_1, and E_2 are then assigned the average values $(T+T_1)/2$, $(T+T_2)/2$, $(\epsilon+\epsilon_1)/2$, and $(\epsilon+\epsilon_2)/2$, respectively, and eqs (37a) and (37b) are solved again for more accurate values of T and ϵ. Another iteration may be performed if desired.

Figure 4b depicts a rhombus shaped mesh formed by the intersection of characteristics. If values of ϵ, w, and T are known at each of points 1, 2, and 3 the values of ϵ, w, and T at the vertex are found by solving the system of equations: eq (21a) which holds along the upper characteristic passing through point 1, eq (10g) which holds along the vertical line passing through point 2, and eq (21b) which holds along the upper characteristic passing through point 3. The solution is found by an iterative process similar to that just described.

By applying successively the methods of solution associated with the mesh types depicted in figures 4a and 4b, values of ϵ, w, and T can be found at each of the mesh points in the domain between the t axis and the characteristic $x=ct$.

The values of α and β at the origin are next calculated from eqs (30a) and (30b). The values of u and v are already known from the boundary conditions. The solution for the mesh situation depicted in figure 4c can then be used to determine the course of the transverse wave front characteristic and the values of ϵ, w, T, u, v, α, and β at its intersections with the other characteristics. In figure 4c the quantities ϵ, w, T, u, v, α, and β are known at point 1, and the values of ϵ, w, and T are known at points 2 and 3. The values of ϵ, w, T, u, v, α, and β are required at the intersection of the transverse wave front characteristic with the characteristic passing through points 2 and 3. As a first step P is set equal to $\sqrt{T_1/m(1+\epsilon_1)}$ and the line $(x-x_1)=P(t-t_1)$ is drawn through point 1 to intersect the characteristic between points 2 and 3. The values of T and ϵ at the intersection are determined by interpolation. The quantity P is then set equal to $\sqrt{\overline{T}/m(1+E)}$, where $\overline{T}$ and E are the averages of T and T_1, and ϵ and ϵ_1 respectively. One or two iterations are performed to obtain more accurate values of T and ϵ and the coordinates (x, t) of the intersection. The value of w at the intersection is then found by interpolation. The values of u, v, α, and β at the intersection are found by solving the set of four equations (30a), (30b), (25a), and (25b) expressed in the form

$$u-u_1-P(\alpha-\alpha_1)=\frac{A}{1+E}[w-w_1-P(\epsilon-\epsilon_1)]$$

$$u+P\alpha=w+P(1+\epsilon)$$

$$v-v_1-P(\beta-\beta_1)=\frac{B}{1+\epsilon}[w-w_1-P(\epsilon-\epsilon_1)]$$

$$v+P\beta=0. \qquad (38a,b,c,d)$$

In the solution process A and B are first set equal to α_1 and β_1 respectively and in subsequent iterations set equal to the average values along the transverse wave front characteristic.

Solutions are needed for two additional mesh situations in order to find the values of u, v, α, and β at the mesh points between the t axis and the transverse wave front characteristic. Figure 4d depicts a mesh of the type found along the t axis. The characteristic between points 1 and 3 may be either the transverse wave front characteristic or one of the solutions of the characteristic equation $dx/dt=c$. Usually the mesh is triangular in shape with point 1 lying on the t axis, but sometimes when the transverse wave front characteristic has a steep slope, the mesh has the more general form shown. The values of ϵ, w, T, u, v, α, β and the coordinates x and t are known at points 1 and 3, and the values of u, v, w, ϵ, and T are known at point 4. The values of α and β at point 4 are required. As a first step the characteristic of slope $-1/P$ is drawn

between points 4 and 2. In this construction iteration and interpolation processes are used to determine values of P, ϵ_2, and T_2, and the coordinates (x_2, t_2) of point 2. Interpolated values for u_2, v_2, α_2, β_2, and w_2 are calculated next. The values of α and β at point 4 are then obtained by solving eqs (26a) and (26b).

Figure 4e depicts a four-sided mesh in which the characteristic between points 3 and 5 may be either the transverse wave front characteristic or a solution of the characteristic equation $dx/dt = c$. Values of u, v, w, α, β, ϵ, and T are known at points 1, 3, and 5, and values of ϵ, w, and T are known at point 6. The values of u, v, α, and β at point 6 are required. To find the solution lines of slope $+1/P$ and $-1/P$ are constructed to intersect at points 2 and 4, respectively, and values of u, v, w, α, β, ϵ, and T are determined at these points. The values of u and α at point 6 are then obtained by solving eqs (25a) and (26a) expressed in the form

$$u - u_2 - P(\alpha - \alpha_2) = \frac{A}{1+E}\left[w_6 - w_2 - P(\epsilon_6 - \epsilon_2)\right]$$

$$u - u_4 + P(\alpha - \alpha_4) = \frac{A}{1+E}\left[w_6 - w_4 + P(\epsilon_6 - \epsilon_4)\right]. \tag{39a,b}$$

The values of v and β at point 6 are obtained similarly by solving eqs (25b) and (26b).

By systematically applying the methods of solution outlined for the five mesh types depicted in figure 4, it is possible to calculate numerical values of ϵ, w, T, u, v, α, and β at any distance x along the filament, and at any time t.

In order to find the configuration of the filament in laboratory (x', y') coordinates it is only necessary to calculate appropriate values of $\xi(x, t)$, and $\eta(x, t)$ by integrating $u(x, t)$ and $v(x, t)$ with respect to time. It is of special interest to perform this calculation in order to determine how much curvature is introduced into the transverse wave portion of the filament by the creep and stress relaxation processes that occur after impact.

The methods of calculation just described have been incorporated into a set of programs for use with a digital computer. Calculations have been made for various values of the propagation velocity c, impact velocity V, relaxation time τ, and parameter λ that approximate conditions realized in tests on real materials. It is planned to present some of the results of these calculations in a later publication.

9. References

[1] D. R. Petterson, G. M. Stewart, F. A. Odell, and R. C. Maheux, Textile Research J. **30,** 411 (1960).
[2] D. R. Petterson and G. M. Stewart, Textile Research J. **30,** 422 (1960).
[3] J. W. Jameson, G. M. Stewart, D. R. Petterson, and F. A. Odell, Textile Research J. **32,** 858 (1962).
[4] J. C. Smith, C. A. Fenstermaker, and P. J. Shouse, Textile Research J. **33,** 919 (1963).
[5] J. C. Smith, C. A. Fenstermaker, and P. J. Shouse, Textile Research J. **35,** 743 (1965).
[6] C. A. Fenstermaker and J. C. Smith, Applied Polymer Symposia **1,** 125 (1965).
[7] Kh. A. Rachmatulin, Prikl. Mat. i. Mech. **9,** 91 (1945).
[8] G. I. Taylor, Scientific Papers of G. I. Taylor, Vol. **1,** Paper No. 30, 456, Cambridge University Press (1958).
[9] G. I. Taylor, Scientific Papers of G. I. Taylor, Vol. **1,** Paper No. 32, 467, Cambridge University Press (1958).
[10] T. von Karman and P. Duwez, J. Appl. Phys. **21,** 987 (1950).
[11] J. C. Smith, F. L. McCrackin, and H. F. Schiefer, J. Res. NBS **60,** 517 (1958); Textile Res. J. **28,** 288 (1958).
[12] J. C. Smith, J. M. Blandford, and K. M. Towne, Textile Res. J. **32,** 67 (1962).
[13] L. E. Malvern, J. Appl. Mech. **18,** 203 (1951).
[14] L. E. Malvern, Quart. Appl. Math. **8,** 405 (1951).
[15] E. H. Lee and I. Kanter, J. Appl. Phys. **24,** 1115 (1953).
[16] R. D. Glauz and E. H. Lee, J. Appl. Phys. **25,** 947 (1954).
[17] J. A. Morrison, Quart. Appl. Math. **14,** 153 (1956).
[18] J. C. Smith, J. Appl. Phys. **37,** 1697 (1966).
[19] N. Cristescu, Arch. Mech. Stos. **12,** 597 (1960).
[20] N. Cristescu, J. Mech. Phys. Solids **9,** 165 (1961).
[21] N. Cristescu, Arch. Mech. Stos. **15,** 47 (1963).
[22] N. Cristescu, International Symposium on Stress Waves in Anelastic Solids, p. 118 (Springer Verlag, 1964).
[23] N. Cristescu, J. Mech. Phys. Solids **12,** 269 (1964).
[24] A. G. Webster, Partial Differential Equations of Mathematical Physics, p. 283, 2d ed. (Dover, 1955).

(Paper 70B4–187)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

Functions for Thermal Stress Calculation Near a Transient Heat Source on a Flat Surface

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(June 7, 1966)

When an initially unstressed elastic solid at uniform temperature is subject to a transient, locally two-dimensional heat flux on a flat surface, the two-dimensional total stress field near the wall is locally determined for short times and may be constructed from the functions described in this report; Fortran programs are available for their computation. In particular, maximum total stress and total stress fields for initially large heat fluxes are readily obtained for estimations of yield probability.

Key Words: Elastic stress, heat transfer, plane strain, thermal stress, yield stress.

1. Introduction

When heat is suddenly applied to or withdrawn from the surface of an elastic solid, large superficial stresses are set up which can cause surface damage when local elastic limits are exceeded. This occurs, for example, when hot or cold material is suddenly brought into contact with the solid surface. If the surface is reasonably flat and the initial heat flux distribution rapidly varying in one direction only (for example, near the perimeter of contact of the hot material mentioned above), the initial transient stresses near the wall are obtainable from plane strain theory.

In this report, plane strain theory is used to analyze the total stress field near the surface of a semi-infinite elastic solid during a short time interval during which a large two-dimensional heat flow occurs; formulae are presented for numerical evaluation of the stress for a class of heat flux pulses which may be approximated by a simple physically relevant form. Such a stress field is useful for estimating the probability and location of fracture from either maximum stress-yield strength data or a more sophisticated probability theory.

We shall consider then a semi-infinite elastic solid occupying the region $y \geq 0$ which at time $t=0$ has vanishing stress components, $\sigma_{ij}=0$, and uniform temperature $T=0$. Cartesian coordinates x and z are taken to lie in the surface and a given *outward* heat flux $q(x, t)$ independent of z is stipulated, so that the temperature $T(x, y, t)$, the stresses $\sigma_{ij}(x, y, t)$, the strains $\epsilon_{ij}(x, y, t)$, and the displacements $u_i(x, y, t)$, $(i=1, 2)$, do not depend on z. We assume also (plane strain) $u_3(x, y, t)=0$.

We shall neglect the heat generated in the solid by the strains, and assume that the temperature is governed by the heat equation

$$T_{,t}=D(T_{,xx}+T_{,yy}), \tag{1}$$

where D is the coefficient of thermal diffusion. We shall also neglect the inertial effects in the solid, assuming that the stress is quasi-stationary near the surface. Then

$$\sigma_{ij,\,j}=0, \qquad (i, j=1, 2, 3), \tag{2}$$

where the stress-stain relation [1][3] is given by

$$\sigma_{ij}=\lambda\delta_{ij}\epsilon_{kk}+2\mu\epsilon_{ij}-mT\delta_{ij} \tag{3}$$

the strains are given in terms of the displacements by:

$$\epsilon_{ij}=\tfrac{1}{2}(u_{i,\,j}+u_{j,\,i}), \qquad (i, j=1, 2, 3). \tag{4}$$

Here λ and μ are the Lame constants, α is the coefficient of linear thermal expansion, and $m=\alpha(3\lambda+2\mu)$. The Lame constants λ and μ are given in terms of the Young's modulus E and Poisson's ratio ν as follows [1]:

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu=\frac{E}{2(1+\nu)}.$$

The validity of the assumptions made in the above equations will depend on the specific heat flux involved, $q(x, t)$, and the region being studied, and should be checked a posteriori.

We shall assume further that a single scalar, the total stress (the first stress invariant),

$$\sigma \equiv \sigma_{ii}=\sigma_{11}+\sigma_{22}+\sigma_{33}, \tag{5}$$

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[3] Figures in brackets indicate the literature references on page 270.

suffices for yield analysis. Yield criteria may be based on extreme values of σ or on the more sophisticated probability theories using the stress distribution [2, 3].

Boundary conditions on the stress system are that surface forces vanish on $y=0$:

$$\sigma_{12}=\sigma_{22}=0, \qquad \text{on } y=0. \tag{6}$$

It is well known [4] that when two bodies at different temperatures are put in contact at $t=0$, heat is transferred at a rate $0(t^{-1/2})$ for sufficiently short times. We shall assume the heat flux time history in a time interval of interest, $0\leq t\leq t_F$, is approximated as follows:

$$q(x, t)=\sum_{n=0}^{3} A_n(x)t^{n-1/2}, \qquad (0\leq t\leq t_F). \tag{7}$$

In particular, we shall find the total stress fields $\sigma_{n\delta}(x, y, t)$ associated with the line source heat fluxes:

$$q_{n\delta}(x, t)=\delta(x)t^{n-1/2}, \qquad (n=0, 1, 2, 3). \tag{8}$$

Then the stress $\sigma(x, y, t)$ for the general heat flux of (7) is given by

$$\sigma(x, y, t)=\sum_{n=0}^{3}\int_{-\infty}^{\infty} ds A_n(s)\sigma_{n\delta}(x-s, y, t). \tag{9}$$

Our principal task is to construct the functions $\sigma_{n\delta}(x, y, t)$.

2. Analytic Evaluation of the Stress $\sigma_{n\delta}(x, y, t)$

For the case of plane strain, the strains and stresses are given in terms of the displacements as follows:

$$\begin{cases} \epsilon_{11}=u_{1,x} \\ \epsilon_{12}=\epsilon_{21}=\frac{1}{2}(u_{1,y}+u_{2,x}) \\ \epsilon_{22}=y_{2,y} \\ \epsilon_{i3}=\epsilon_{3i}=0 \qquad (i=1, 2, 3) \\ \epsilon_{ii}=u_{1,x}+u_{2,y} \end{cases} \tag{10}$$

$$\begin{cases} \sigma_{11}=\lambda\epsilon_{ii}+2\mu u_{1,x}-mT \\ \sigma_{12}=\sigma_{21}=\mu(u_{1,y}+u_{2,x}) \\ \sigma_{22}=\lambda\epsilon_{ii}+2\mu u_{2,y}-mT \\ \sigma_{33}=\lambda\epsilon_{ii}-mT \\ \sigma_{23}=\sigma_{32}=\sigma_{13}=\sigma_{31}=0 \\ \sigma=\sigma_{ii}=(3\lambda+2\mu)\epsilon_{ii}-3mT. \end{cases} \tag{11}$$

The equations (2) become:

$$\begin{aligned} &(\lambda+2\mu)u_{1,xx}+\lambda u_{2,xy}-mT_{,x}+\mu(u_{1,yy}+u_{2,xy})=0 \\ &(\lambda+2\mu)u_{2,yy}+\lambda u_{1,xy}-mT_{,y}+\mu(u_{1,xy}+u_{2,xx})=0. \end{aligned} \tag{12}$$

For a function $f(x, y, t)$, satisfying appropriate conditions, we introduce the Laplace transform $\mathscr{L}$ in t and the Fourier transform $\mathscr{F}$ in x:

$$\begin{aligned} F(y; k, s)&\equiv \mathscr{F}\mathscr{L}f(x, y, t) \\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx e^{-ikx}\int_{0}^{\infty} dt e^{-st}f(x, y, t) \end{aligned} \tag{13}$$

with the inverse:

$$\begin{aligned} f(x, y, t)&=\mathscr{F}^{-1}\mathscr{L}^{-1}F(y; k, s) \\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk e^{ikx} \\ &\quad\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds e^{st}F(y; k, s). \end{aligned} \tag{14}$$

The transforms of (u_1, u_2, T, σ) are denoted, respectively, by (U, V, Θ, Σ).

We obtain the system of ordinary differential equations:

$$\begin{cases} (s+k^2D)\Theta-D\Theta_{,yy}=0 \\ -(\lambda+2\mu)k^2U+\mu U_{,yy}+(\lambda+\mu)ikV_{,y}-ikm\Theta=0, \\ -\mu k^2V+(\lambda+2\mu)V_{,yy}+(\lambda+\mu)ikU_{,y}-m\Theta_{,y}=0. \end{cases} \tag{15}$$

Boundary conditions are boundedness of the transforms as $y\to\infty$, and the following conditions at $y=0$ follow from (6):

$$U_{,y}+ikV=0, \qquad ik\lambda U+(\lambda+2\mu)V_{,y}-m\Theta=0, \tag{16.1}$$

and by (8) at $y=0$:

$$K\Theta_{,y}=\frac{1}{\sqrt{2\pi}}\frac{\Gamma(n+\frac{1}{2})}{s^{n+1/2}}, \tag{16.2}$$

where K is the coefficient of heat conduction. The system of (15) has solutions of the form

$$\begin{pmatrix}\Theta\\ U\\ V\end{pmatrix}=\begin{pmatrix}\Theta_1\\ U_1\\ V_1\end{pmatrix}e^{-\nu_1 y}+\begin{pmatrix}0\\ U_2+yU_3\\ V_2+yV_3\end{pmatrix}e^{-ky}, \tag{17}$$

where:

$$\nu_1 = \left(k^2 + \frac{s}{D}\right)^{1/2},$$
$$\epsilon = \operatorname{sgn} k, \tag{18}$$

and ν_1 has nonnegative real part. Substitution of (17) into (15) and use of the conditions (16.1) gives the following:

$$\begin{cases} U = [q_{11}e^{-\nu_1 y} + (q_{12} + yq_{13})e^{-|k|y}]\Theta_1 \\ V = [q_{21}e^{-\nu_1 y} + (q_{22} + yq_{23})e^{-|k|y}]\Theta_1, \end{cases} \tag{19}$$

where:

$$q_{11} = \frac{ikmD}{(\lambda + 2\mu)s},$$

$$q_{21} = -\frac{\nu_1 mD}{(\lambda + 2\mu)s},$$

$$q_{12} = \frac{mD}{2iks(\lambda + \mu)(\lambda + 2\mu)}\{2\epsilon(\lambda + 2\mu)k\nu_1 - 2\mu k^2\},$$
$$q_{13} = \frac{imD}{2\mu s(\lambda + 2\mu)}\{2\mu k\nu_1 - 2\mu k^2\epsilon\}, \tag{20}$$

$$q_{22} = \frac{mD}{2k\mu s(\lambda + \mu)(\lambda + 2\mu)}\{-2\mu^2 k\nu_1 + 2\mu(\lambda + 2\mu)k^2\epsilon\},$$

$$q_{23} = i\epsilon q_{13}.$$

The stress transform Σ is readily found to contain no term linear in y:

$$\Sigma = \left[\alpha_1 e^{-\nu_1 y} + \left(\alpha_2 \frac{\nu_1|k|}{s} + \alpha_3 \frac{k^2}{s}\right)e^{-|k|y}\right]\Theta_1 \tag{21}$$

where the α_i do not depend on (k, s, y):

$$\alpha_1 = -\frac{4\mu m}{\lambda + 2\mu}$$

$$\alpha_2 = -\alpha_3 = \frac{2\mu mD(3\lambda + 2\mu)}{(\lambda + \mu)(\lambda + 2\mu)}. \tag{22}$$

We notice that the first term on the right-hand side of (21) gives a term directly proportional to the temperature (17). By (16.2) and (17),

$$\Theta_1 = -\frac{1}{K\sqrt{2\pi}}\frac{\Gamma(n + \frac{1}{2})}{\nu_1 s^{n+1/2}}. \tag{23}$$

Inverse transforms of the terms in (21) may be obtained by manipulation of tabulated transforms. It is convenient to introduce the variables:

$$r = \sqrt{x^2 + y^2}, \qquad (\xi, \eta, \rho) = \frac{(x, y, r)}{\sqrt{4Dt}},$$
$$\varphi = \tan^{-1}\frac{\xi}{\eta}, \qquad z = \rho^2 e^{-2i\varphi}, \qquad |\arg z| \leq \pi. \tag{24}$$

Then for the first of the terms in (21), we note [5a]:

$$\mathcal{L}^{-1}\mathcal{F}^{-1}(e^{-\nu_1 y}/\nu_1 s^{n+1/2})$$
$$= \frac{1}{\sqrt{2\pi}} t^{n-1/2} e^{-|z|}\psi\left(\frac{1}{2} + n, 1; |z|\right), \tag{25}$$

where we have made use of the relation between the Whittaker function $W_{-n,0}(z)$ and the confluent hypergeometric [6a] function $\psi(\frac{1}{2} + n, 1; z)$.

$$\mathcal{L}^{-1}\mathcal{F}^{-1}\frac{|k|e^{-|k|y}}{s^{n+3/2}} = \frac{1}{\sqrt{2\pi D}}\frac{t^{n-1/2}}{\Gamma\left(n + \frac{3}{2}\right)}\frac{\cos 2\varphi}{|z|} \tag{26}$$

For the third term of (21) we note that

$$\mathcal{L}^{-1}\mathcal{F}^{-1}\frac{k^2 e^{-|k|y}}{\nu_1 s^{n+3/2}} = \sqrt{\frac{2}{\pi}}\frac{\partial^2}{\partial y^2} I_n(x, y, t), \tag{27}$$

where:

$$I_n(x, y, t) \equiv \mathcal{L}^{-1}\frac{1}{s^{n+3/2}} J(x, y, s),$$
$$J(x, y, s) \equiv \int_0^\infty dk\ \frac{\cos kx \cdot e^{-ky}}{\sqrt{k^2 + \frac{s}{D}}}. \tag{28}$$

Then in terms of Struve's function $\mathbf{H}_0(z)$ and Bessel's function [5c] of the second kind $Y_0(z)$,

$$J(x, y, s) = \frac{\pi}{4}\left[\mathbf{H}_0\left(re^{-i\varphi}\sqrt{\frac{s}{D}}\right) - Y_0\left(re^{-i\varphi}\sqrt{\frac{s}{D}}\right)\right]$$
$$+ \frac{\pi}{4}\left[\mathbf{H}_0\left(re^{i\varphi}\sqrt{\frac{s}{D}}\right) - Y_0\left(re^{i\varphi}\sqrt{\frac{s}{D}}\right)\right]. \tag{29}$$

Then [5d, 6b],

$$I_{-1}(x, y, t) = \frac{1}{2\sqrt{\pi t}}\operatorname{Real}\left\{e^{z/2}K_0\left(\frac{z}{2}\right)\right\}$$
$$= \frac{1}{2\sqrt{t}}\operatorname{Real}\left\{\psi\left(\frac{1}{2}, 1; z\right)\right\}, \tag{30}$$

where K_0 is the modified Bessel function of the second

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kind. It follows that [5e]:

$$I_n(x, y, t) = \left(\int_0^t dt \cdot \right)^{n+1} I_{-1}(x, y, t)$$

$$= \frac{1}{2}(-)^{n+1} \left(\frac{r^2}{4D}\right)^{n+1/2} \text{Real} \left\{ e^{-i\varphi(2n+1)} \cdot \left(\int_\infty^z \frac{dz}{z^2} \right)^{n+1} z^{1/2} \psi\left(\frac{1}{2}, 1; z\right) \right\} \quad (31)$$

where $(\int dt \cdot)^{n+1}$ is the $(n+1)$-th iterated integral. Each integral is independent of the argument of the lower limit for z if $|\arg z| < 3\pi/2$. It is easy to establish by recursion the following formula [6e]:

$$\left(\int_{-\infty}^z \frac{dz}{z^2} \cdot \right)^{n+1} z^{1/2} \psi\left(\frac{1}{2}, 1; z\right) = \frac{C_n^*}{z^{n+1/2}} \psi\left(-n-\frac{1}{2}, 1; z\right) - \sum_{K=0}^{n} \frac{(-)^K}{z^K} \frac{C_{n-K}^*}{K!}, \quad C_n^* \equiv \left[\frac{\pi}{\Gamma\left(n+\frac{3}{2}\right)} \right]^2$$

The finite sum removes from the first term of the right-hand side those terms of the asymptotic series which would give a nonvanishing result as $z \to \infty$ [6d].

We introduce the following real-valued functions:

$$G_{1n}(\xi, \eta) = \text{Real} \left\{ \psi\left(\frac{1}{2} - n, 1; z\right) \right\},$$

$$G_{2n}(\xi, \eta) = \text{Real} \left\{ (4n^2-1) z \psi\left(\frac{3}{2} - n, 3; z\right) + (2n+1) \psi\left(\frac{1}{2} - n, 2; z\right) \right\}. \quad (33)$$

Finally then we are able to write the total stress as follows:

$$\sigma_{n\delta}(x, y, t) = \frac{m}{K} t^{n-1/2} \{ \Lambda_1 f_n(\xi, \eta) + \Lambda_2 h_n(\xi, \eta) \} \quad (34)$$

where the functions (f_n, h_n) are even in ξ and do not depend on the material parameters:

$$f_n(\xi, \eta) = -\frac{1}{\pi} \frac{2}{2n+1} \frac{\cos 2\varphi}{\rho^2} + \frac{(-)^{n+1}}{(2n+1)\Gamma\left(n+\frac{3}{2}\right)} \cdot \left\{ G_{2n}(\xi, \eta) - \sum_{K=0}^{n} \frac{(-)^K}{K!} \left[\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n-K+\frac{3}{2}\right)} \right]^2 \cdot (2n+1-2K)(2n-2K) \rho^{2n-1-2K} \cos \varphi\, (2n-1-2k) \right\}, \quad (35.1)$$

$$h_n(\xi, \eta) = \frac{\Gamma(n+\frac{1}{2})}{\pi} e^{-\rho^2} \psi(n+\tfrac{1}{2}, 1; \rho^2). \quad (35.2)$$

The constants Λ_i in (34) are the following:

$$\Lambda_1 = \frac{\mu(3\lambda+2\mu)}{2(\lambda+\mu)(\lambda+2\mu)}, \qquad \Lambda_2 = \frac{2\mu}{(\lambda+2\mu)}. \quad (36)$$

We note the useful relation:

$$T_{n\delta}(x, y, t) = -\frac{1}{2K} t^{n-1/2} h_n(\xi, \eta), \quad (37)$$

giving the temperature field in the solid due to the line heat source of (8).

3. Numerical Evaluation of (f_n, h_n)

Expressions are given in appendix A for the interim functions $G_{1n}(\xi, \eta)$, $G_{2n}(\xi, \eta)$, both convergent and asymptotic series. Using double precision computation (25 digit mantissa) for the convergent series, single precision computation (10 digit mantissa, exponent less than 308) for the asymptotic series, agreement between the two was to six significant figures at $\rho^2 = 20$ for several values of φ on $0 \leq \varphi \leq \frac{\pi}{2}$, and thus $\rho = \sqrt{20}$ was taken as the crossover point.

Expressions for $h_n(\xi, \eta)$ are also given in appendix A in terms of the functions $G_{1n}(0, \rho)$ using a recursion relation.

As $\rho \to 0$, the G_{2n} have integrable singularities:

$$f_n \to \frac{1}{\pi} \left\{ \ln \rho^2 + \left(\psi\left(\frac{1}{2} - n\right) - \psi(1) - \psi(2) + 2 \right) + 0(\rho) \right\},$$

$$h_n \to -\frac{1}{\pi} \left\{ \ln \rho^2 + \left(\psi\left(\frac{1}{2} + n\right) - 2\psi(1) \right) + 0(\rho) \right\}, \quad (38)$$

where $0(\rho)$ vanishes as $\rho \to 0$, and $\psi(x)$ is the Digamma function [7]. We may then integrate $\sigma_{n\delta}$ over $y = 0$ as required in the convolution (9).

For large ρ, the simple forms,

$$f_n \sim -\frac{1}{\pi} \frac{2}{2n+1} \frac{\cos 2\varphi}{\rho^2} + \frac{\Gamma\left(n+\frac{1}{2}\right)}{\pi\Gamma(n+2)} \frac{\cos 3\varphi}{\rho^3} + 0(\rho^{-5}),$$

$$h_n \sim 0, \quad (39)$$

give better than one percent accuracy for $\rho \geq 10$. These formulae show that

$$\sigma_{n\delta} \sim \frac{1}{\rho}\left[\frac{m\Lambda_2}{K}\frac{\Gamma\left(n+\frac{1}{2}\right)\sqrt{4D}}{\pi\Gamma(n+1)}\right]\cdot\frac{t^n y}{x^2+y^2}, \tag{40}$$

so that for short times ($r/\sqrt{4Dt}\to\infty$), the stress field of a line source dies off $0(1/x^2)$ for $|x| \gg y$ (although only $0(1/y)$ for $y \gg |x|$). This integrable decay rate in x makes the convolution (9) useful since the large stresses expected near the surface at early times are locally induced, and the plane stress theory is applicable in more general situations.

A subroutine for the calculation of $\{G_{1n}(\xi, \eta), G_{2n}(\xi, \eta)\}$, and a program employing this subroutine for the calculation of $\{f_n(\xi, \eta), h_n(\xi, \eta)\}$ for a given value of η on the set of values,

$$\xi = 0.0(0.1)10.0, \tag{41}$$

is discussed in appendix B. The calculated values are stored on tape in consecutive values of ξ in the following order for each ξ:

$$\{f_0(\xi, \eta), \ldots, f_3(\xi, \eta), h_0(\xi, \eta), \ldots, h_3(\xi, \eta)\}. \tag{42}$$

The values are sufficiently slowly varying that Simpson's rule integration in (9) is normally adequate. Calculation of these functions on the set (41) for a given value of η takes on the average about two minutes, and was done for the set of integer values

$$\eta = 0.0(1.0)10.0. \tag{43}$$

These data are now on magnetic tape, and are available on punched cards. (The singular values ($\xi = 0$, $\eta = 0$) have been replaced by the values at ($\xi = 0.1$, $\eta = 0$) as being adequate for integration.)

4. Step-Input Stress Response

A special case of heat flux of particular interest for stress evaluation near the perimeter of an (almost) uniform source is obtained from the general source of (7) by setting

$$A^n(x) = A_n h(x), \tag{44}$$

where the A_n are constants and

$$h(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0). \end{cases} \tag{45}$$

Then the resulting stress σ_h is found by (9) to be

$$\sigma_h(x, y, t) = \frac{m\sqrt{4D}}{K}\sum_{n=0}^{3} A_n t^n\{\Lambda_1 F_n(\xi, \eta) + \Lambda_3 H_n\ (\xi, \eta)\}, \tag{46}$$

where:

$$\begin{pmatrix} F_n \\ H_n \end{pmatrix} \equiv \int_{-\infty}^{\xi} ds \begin{pmatrix} f_n(|s|, \eta) \\ h_n(|s|, \eta) \end{pmatrix}, \tag{47}$$

$$(\xi, \eta) = \frac{(x, y)}{\sqrt{4Dt}}.$$

When the entire path of integration (ξ_1, ξ_2) lies outside the circle of radius $\rho = 10$, the asymptotic forms (39) are used to evaluate:

$$\phi_n(\xi_2, \xi_1; \eta) = \int_{\xi_1}^{\xi_2} ds\, f_n(|s|, \eta), \tag{48}$$

each term of (39) having a simple closed-form integral. Then for $0 \leq \eta \leq 10$, we have

$\xi \leq -10.$:

$$\begin{pmatrix} F_n(\xi, \eta) \\ H_n(\xi, \eta) \end{pmatrix} = \begin{pmatrix} \phi_n(\xi, -\infty; \eta) \\ 0 \end{pmatrix}; \tag{49}$$

$-10. \leq \xi \leq 10.$:

$$\begin{pmatrix} F_n(\xi, \eta) \\ H_n(\xi, \eta) \end{pmatrix} = \begin{pmatrix} F_n(-10, \eta) \\ 0 \end{pmatrix} + \int_{-10}^{\xi} ds \begin{pmatrix} f_n(|s|, \eta) \\ h_n(|s|, \eta) \end{pmatrix}; \tag{50}$$

$\xi \geq 10.$:

$$\begin{pmatrix} F_n(\xi, \eta) \\ H_n(\xi, \eta) \end{pmatrix} = \begin{pmatrix} F_n(10, \eta) \\ H_n(10, \eta) \end{pmatrix} + \begin{pmatrix} \phi_n(\xi, 10; \eta) \\ 0 \end{pmatrix}. \tag{51}$$

for $\eta > 10$, (49) is valid for all ξ.

Appendix C describes a program for determining $\{F_n, H_n\}$ for given η, (one of the set (43) of course) on the set

$$\xi = 10.(0.2)20., \tag{52}$$

using Simpson's rule for the evaluation of the integrals in (50). These results are on magnetic tape and are available on punched cards. Calculations took about one-half minute for each value of η.

When $A_0 = 1$, $A_1 = A_2 = A_3 = 0$, the total stress field is self-similar in time, dependent ξ and η only, and rises monotonically (dominated by $G_n(\xi, \eta)$) in ξ to a maximum value at $\xi = \infty$ independent of η. This value is attained at ($\eta = 0$, $\xi = 10$), to the accuracy of the calculation, and may be determined from the values

$$\begin{pmatrix} F_0(20, 0) \\ H_0(20, 0) \end{pmatrix} = \begin{pmatrix} -0.0637 \\ 1.7724 \end{pmatrix}. \tag{53}$$

For large values of ξ, the stress field σ_h given by (46) is uniform in x for any coefficient set $\{A_n\}$, and agree with the results for heat flux $q(t)$ uniform in x described in appendix D.

5. Conclusions

Formulae are presented for the estimation of total stress fields near the surface of a solid during a short time interval during which a large heat transfer occurs. Such a stress field is useful for estimating the probability and location of fracture, either from maximum stress-yield strength data or from a more sophisticated probability theory. The stress fields obtained may be expected to be valid in a region adjacent to a point on the solid surface of dimension $l \approx \sqrt{4Dt}$ and for the time interval $0 \leq t \leq t_F$, the period of validity of the heat flux approximation (7), so long as the following orders hold:

$$\sqrt{4Dt} \ll \{L, R_m, R_q\}. \tag{54}$$

where L is a length scale for the solid, and R_m and R_q are, respectively, the mean radius of curvature of the solid surface and the radius of curvature of lines of constant outward heat flux at the point in question.

It would be desirable to have a "maximum principle" available, stating, for example, that if $q(x, t) = 0$ for $t > t_F$, then the maximum value of $\sigma(x, y, t)$ over (x, y) for any $t > t_F$ does not exceed the maximum value of $\sigma(x, y, t_F)$ over (x, y). The part of the stress associated with $h_n(\xi, \eta)$ shares with the temperature field given by (37) such a maximum principle, but no more general result appears available.

The authors thank Robert Arnett of the Boulder Laboratories, National Bureau of Standards, for suggesting the problem and for many valuable comments.

6. References

[1] B. A. Boley and J. H. Weiner, Theory of Thermal Stresses (John Wiley & Sons, New York, N.Y., 1960).

[2] W. Weibull, A Statistical Theory of the Strength of Materials, Proc. Royal Swedish Inst. Eng. Res., 151 (1939).

[3] J. Tucker, Jr., Statistical Theory of the Effect of Divisions and Method of Loading Upon the Modulus of Rupture of Beams, Proc. ASTM, **41**, pp. 1072–1094 (1941).

[4] H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2d edition (Oxford, Clarendon Press, 1959).

[5] A. Erdelyi, Tables of Integral Transforms **V.** 1, Bateman Manuscript Project (McGraw-Hill Book Co., Inc., New York, N.Y., 1954); (a) p. 283, #43; (b) p. 15, #7; (c) p. 138, #10; (d) p. 287, #14 (#12 is in error); (e) p. 130, #9.

[6] A. Erdelyi, Higher Transcendental Functions, **V.** 1, Bateman Manuscript Project (McGraw-Hill Book Co., Inc., New York, N.Y., 1953); (a) p. 264, 6.9(2); (b) p. 265, 6.9.1(13); (c) p. 261, 6.7.1(13); (d) p. 278, 6.13.1(1); (e) p. 258, 6.6(13).

[7] M. Abramowitz and L. A. Stegun, Handbook of Mathematical Functions, AMS **55**, p. 258, National Bureau of Standards (1964).

7. Appendix A. The Functions $\{G_{1n}(\xi, \eta), G_{2n}(\xi, \eta), h_n(\xi, \eta)\}$

The following expansions for G_{1n}, G_{2n} are readily obtained from the definitions of (33) and the properties [6c] of the confluent hypergeometric functions $\psi(a, b; z)$:

$$G_{1n}(\xi, \eta) = \frac{-1}{\Gamma\left(\frac{1}{2}-n\right)} \left\{ \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \frac{\Gamma\left(\frac{1}{2}-n+r\right)}{\Gamma\left(\frac{1}{2}-n\right)} \cdot \rho^{2r} [(\ln \rho^2) \cos 2r\varphi - 2\varphi \sin 2r\varphi] \right.$$

$$\left. + \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}-n-r\right)}{\Gamma\left(\frac{1}{2}-n\right)} \frac{1}{(r!)^2} \left[\psi\left(\frac{1}{2}-n+r\right) - 2\psi(1+r)\right] \cdot \rho^{2r} \cos 2r\varphi \right\}, \tag{A1}$$

$$G_{1n}(\xi, \eta) \sim \rho \sum_{m=0}^{\infty} (-)^m \frac{\rho^{2n-1-2m}}{m!} \cos \varphi(2n-1-2m) \cdot \left[\frac{\Gamma\left(m+\frac{1}{2}-n\right)}{\Gamma\left(\frac{1}{2}-n\right)}\right]^2, \left(|\varphi| \leq \frac{\pi}{2}\right). \tag{A2}$$

$$G_{2n}(\xi, \eta) = \frac{(-)^{n+1}}{\pi} \Gamma\left(n+\frac{3}{2}\right) \left\{ (2n+1) \ln \rho^2 + (1+2n)\left(\psi\left(\frac{1}{2}-n\right) - \psi(1) - \psi(2) + 2\right) + \frac{2\cos 2\varphi}{\rho^2} \right.$$

$$+ (1+2n) \sum_{l=1}^{\infty} \frac{(1+2l)}{l!} \frac{\Gamma\left(\frac{1}{2}-n+l\right)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma(l+2)} \rho^{2l} \cdot [(\ln \rho^2) \cos 2l\varphi - 2\varphi \sin 2l\varphi]$$

$$+ (1+2n) \sum_{l=1}^{\infty} \frac{\Gamma\left(\frac{1}{2}-n+l\right)}{l!\Gamma(l+2)\Gamma\left(\frac{1}{2}-n\right)} \cdot \left[2 + (2l+1) \cdot \left(\psi\left(\frac{1}{2}-n+l\right) - \psi(1+l) - \psi(2+l)\right)\right] \rho^{2l} \cos 2l\varphi \left. \right\}, \tag{A3}$$

$$G_{2n}(\xi,\eta) \sim \sum_{m=0}^{\infty} \frac{(-)^m}{m!}\left[\frac{\Gamma\left(-\frac{1}{2}-n+m\right)}{\Gamma\left(-\frac{1}{2}-n\right)}\right]^2$$

$$(2n+1-2m)(2n-2m)\cdot\rho^{2n-1-2m}\cos\varphi(2n-1-2m),$$

$$\left(|\varphi|\leq\frac{\pi}{2}\right). \qquad \text{(A4)}$$

Let

$$h_n^*(\xi,\eta)\equiv\frac{\Gamma\left(n+\frac{1}{2}\right)}{\pi}e^{-\rho^2}G_{1n}(0,\rho),$$

$$(n=1,2). \qquad \text{(A5)}$$

Then $h_n(\xi,\eta)$ are found from the following expressions:

$$h_0=h_0^*,$$

$$h_1=2\rho^2h_0^*-4h_1^*,$$

$$h_2=\frac{1}{3}[4\rho^2(2+\rho^2)-1]h_0^*$$

$$-\frac{8}{3}(2+\rho^2)h_1^*, \qquad \text{(A6)}$$

$$h_3=\frac{3}{10}\left\{\left[\frac{4}{9}(4+\rho^2)(4\rho^2(2+\rho^2)-1)-4\rho^2\right]h_0^*\right.$$

$$\left.-8\left[\frac{4}{9}(4+\rho^2)(2+\rho^2)-1\right]h_1^*.\right.$$

8. Appendix B. Description and Use of Fortran Programs for Calculation of $\{f_n, h_n\}$

From the expansions for G_{1n} and G_{2n}, given in appendix A, it can be seen that there are multiplicative factors in the sums which are dependent only on n and the index of summation. These factors were therefore computed using the program GCOEFFS and were punched on cards in a format suitable for use directly as part of an object deck in another program. It was decided that the maximum number of terms necessary for required accuracy is 20 for the asymptotic expansions and 100 for the convergent series.

Computation of the functions $\{f_n(\xi,\eta), h_n(\xi,\eta)\}$ is done in program FORMFG using subroutines GCOEF1, GCOEF2, FSGSHS, and GSUMS written in CDC 3600 Fortran. The main program FORMFG provides the input values ξ and η and writes the computed functions on a tape. Subroutine FSGSHS performs the actual computation of the functions (f_n, h_n) for each value of (ξ,η). Subroutine GSUMS computes the functions $\{G_{1n}(\xi,\eta), G_{2n}(\xi,\eta)\}$ for each (ξ,η) using the coefficients of G_{1n} and G_{2n} as computed in the program GCOEFFS. Subroutines GCOEF1 and GCOEF2 set up tables of these coefficients for use by GSUMS.

The first input variable, ISKIP, is used to position the output tape, giving the number of files to be skipped before any writing is done on that tape. Since the functions $\{f_n, h_n\}$ are computed for $\xi=-10.(0.1)10.$ for each η, it is necessary to read in only the variable η(DETA). The program will read in an η, do the computations and return to read the next η, continuing until no cards remain. Note that η is read in as a double precision variable.

Program listings for the CDC 3600 may be obtained from the authors.

9. Appendix C. Description and Use of Fortran Programs for Calculation of $\{F_n, H_n\}$

The program INTEGFGH computes the functions $\{F_n, H_n\}$ using the formulas (49), (50), (51). The functions $\{f_n, h_n\}$ are read from the tape generated by program FORMFG. Results are written on an outtape in the same order as are the $\{f_n, h_n\}$.

Input to this program is in two parts: the tape generated by FORMFG, and cards. The first card input variable is KSKIP, which positions the input tape by skipping KSKIP files. This option makes it possible to select the value of η for which the computations are to be performed. The upper limit of ξ(XIF) and an approximation to $-\infty$(AA) are also card input parameters. After each complete integration the input tape is rewound and the next value of KSKIP is read in, followed by the upper limit of ξ and the approximation to $-\infty$. The program continues until no cards remain to be read.

Program listings for the CDC 3600 may be obtained from the authors.

Appendix D. Heat Flux Uniform in x

When the heat flux of the wall in independent of x, the stress field will be also. Simple results may be obtained for this case, and these results have been used to corroborate the numerical results described in the text.

Putting $k=0$ in (21) gives the x-independent case for the Laplace transforms

$$\sum(y,s)=\alpha_1 e^{-y\sqrt{\frac{s}{D}}}\Theta_1(s). \qquad \text{(D1)}$$

If the outward heat flux $q(t)$ has Laplace transform

$Q(s)$, we have

$$Q(s) = -K\sqrt{\frac{s}{D}}\,\Theta_1(s),$$

so

$$\sum(y, s) = \frac{\alpha_1 e}{-K\sqrt{\frac{s}{D}}}\,Q(s).$$

Using the convolution theorem for Laplace transforms, we may obtain the following formula:

$$\sigma(y, t) = -2\sqrt{\frac{t}{\pi}}\,A\int_0^1 dp\,q(t(1-p^2))\,e^{-\frac{n^2}{p^2}}. \quad \text{(D2)}$$

where

$$A = \frac{\sqrt{D}}{K}\,\alpha_1. \quad \text{(D3)}$$

A useful relation can be derived for the wall stress $\sigma(0, t)$ in terms of the wall temperature $T(0, t)$ only, independent of the heat flux history. Setting $y=0$ in (D1) leads to the relation

$$\sigma(0, t) = \alpha_1 T(0, t). \quad \text{(D4)}$$

The relations (D2) and (D4) are preserved by numerical tests of (46).

(Paper 70B4–188)

JOURNAL OF RESEARCH of the National Bureau of Standards – B. Mathematics and Mathematical Physics
Vol. 70B, No. 4, October–December 1966

Publications of the National Bureau of Standards*

Selected Abstracts

Inequalities for the permanental minors of non-negative matrices, R. A. Brualdi and M. Newman, *Can. J. Math.* **18,** *608–615 (1966).*
It is shown that if A is an $n \times n$ non-negative substochastic matrix then the r^{th} permanental compound of A is also substochastic. If in addition A is doubly stochastic, then it is shown that the sum of all principal permanental minors of order r does not exceed $\binom{n-1}{r}+\binom{n-1}{r-1}$ per (A), with equality if and only if A is the identity matrix.

Real two-dimensional representations of the free product of two finite cyclic groups, J. Lehner and M. Newman, *Proc. Camb. Phil. Soc.* **62,** *135–141 (1966).*
All real discrete representations of the free product of two finite cyclic groups by a group of linear fractional transformations are determined.

Congruence subgroups of positive genus of the modular group, M. Newman and M. Knopp, *Ill. J. Math.* **9,** *No. 4, 577–583 (Nov. 1965).*
Let $\Gamma(n)$ be the principal congruence subgroup of Γ of level n, Γ_n the subgroup of Γ generated by $\Gamma(n)$ and $S=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, and $\Gamma_0(n)$ the subgroup of Γ consisting of all elements $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ of Γ such that $c \equiv 0 \pmod{n}$. Then it is proved that if G is a subgroup of Γ containing Γ_n, then either $G=\Gamma$ or $G \subset \Gamma_0(d)$, $d \mid n$, $d > 1$. This is used to prove that a free congruence subgroup of Γ of level n (i.e., a free subgroup of Γ containing $\Gamma(n)$) is of positive genus, provided that $(n, 2.3.5.7.13)=1$.

Scattering properties of concentric soot-water spheres for visible and infrared light, R. W. Fenn and H. Oser, *J. Appl. Opt.* **4,** *1504–1509 (Nov. 1965).*
Based on the theory by Aden and Kerker, computations of the scattering and absorption properties for concentric spherical water-and-particles have been performed for visible and infrared wave lengths. Computations were performed for size parameter values up to 250. Results indicate that, for compound particles with a nucleus smaller than about one-tenth of the total diameter of the particle, the optical properties are almost completely determined by the outer shell. Some results of the computations are presented in graphical form for the scattering and absorption efficience factors and the angular scattering functions.
The frequency dependence of the complex dielectric constant is not taken into account in the actual computations: absorption, scattering and total extinction are given for constant index of refraction. The theory does allow to take this dependence into account, however, if one wishes to do so.

Other NBS Publications

J. Res. NBS 70A (Phys. and Chem.), No. 4 (July–Aug. 1966), $1.00.
Elastic constants of synthetic single crystal corundum. W. E. Tefft.
Phase equilibria as related to crystal structure in the system niobium pentoxide-tungsten trioxide. R. S. Roth and J. L. Waring.
Properties of aqueous mixtures of pure salts. Thermodynamics of the ternary system: water-calcium chloride-magnesium chloride at 25 °C. R. A. Robinson and V. E. Bower.
Properties of aqueous mixtures of pure salts. Thermodynamics of the ternary system: water-sodium chloride-calcium chloride at 25 °C. R. A. Robinson and V. E. Bower.
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J. Res. NBS 70A (Phys. and Chem.), No. 5 (Sept.–Oct. 1966), $1.00.
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J. Res. NBS 70C (Engr. and Instr.), No. 4 (Oct.–Dec. 1966), 75 cents.
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Dimensional metrology. Subject-classified with abstracts, I. H. Fullmer, Misc. Publ. 265 (Aug. 1, 1966), $3.00.
Periodicals and serials received in the Library of the National Bureau of Standards as of October 1965, N. J. Hopper, NBS Misc. Publ. 274 (July 1, 1966), 50 cents. Supersedes Mono. 57.
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Quarterly radio noise data March, April, May 1965, W. Q. Crichlow, R. T. Disney, and M. A. Jenkins, NBS Tech. Note 18–26 (May 2, 1966), 45 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now the Environmental Science Services Administration, Boulder, Colo.)

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The viscosity and thermal conductivity coefficients of dilute argon between 100 and 2000 °K, H. J. M. Hanley, NBS Tech. Note 333 (March 11, 1966), 25 cents.

An introduction to sampled data and switching logic, T. L. Davis, NBS Tech. Note 334 (Apr. 1, 1966), 40 cents. (Formerly Central Radio Propagation Laboratory of the National Bureau of Standards, now Environmental Science Services Administration, Boulder, Colo.)

Zonal harmonics in low frequency terrestrial radio wave propagation, J. R. Johler, NBS Tech. Note 335 (Apr. 13, 1966), 25 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now the Environmental Science Services Administration, Boulder, Colo.)

Calculation of the admittance of a parallel plate capacitor containing a toroid-shaped sample, E. G. Johnson, Jr., NBS Tech. Note 336 (Apr. 21, 1966), 30 cents.

Advances in ionospheric mapping by numerical methods, W. B. Jones, R. P. Graham, and M. Leftin, NBS Tech. Note 337 (May 12, 1966), 45 cents. (Formerly the Central Radio Propagation Laboratory of the National Bureau of Standards, now the Environmental Science Services Administration, Boulder, Colo.)

Acid-base behavior in methanol-water solvents, R. G. Bates and R. A. Robinson (Proc. Intern. Symp. Electrochemical Society, Toronto, Canada, May 4–6, 1964), Book, Chemical Physics of Ionic Solution, ch. 12, pp. 211–235 (John Wiley and Sons, Inc., New York, N.Y., 1966).

Combustion and reaction calorimetry of several compounds of interest in a light element program, G. T. Armstrong (Proc. 3d Meeting of the ICRPG Working Group on Thermochemistry, El Segundo, Calif., Mar. 17–18, 1965), Chemical Propulsion Information Agency **1,** No. 82 (U), 81–88 (July 1965).

Composition, work and interrelation of international and national organization engaged in the standardization of dental materials and therapeutic agents, G. C. Paffenbarger and M. Kumpula, Intern. Dent. J. **15,** No. 4, 571–587 (Dec. 1965).

Crystal imperfections in magnesium fluoride, D. J. Barber, J. Appl. Phys. **36,** No. 10, 3342–3349 (Oct. 1965).

Dental application of polymers: a review, G. M. Brauer, J. Am. Dental Assoc. **72,** 1151–1158 (May 1966).

Dislocations and stacking faults in rutile crystals grown by flame-fusion methods, D. J. Barber and E. N. Farabaugh, J. Appl. Phys. **36,** No. 9, 2803–2806 (Sept. 1965).

Effect of reactions with the atmosphere during fatigue of metals, J. A. Bennett (Proc. Sagamore Conf. Fatigue, Sagamore, New York, Aug. 15, 1963), Book, Fatigue: An Interdisciplinary Approach, pp. 209–227 (Syracuse University Press, Syracuse, N.Y., 1964).

Electrometric method of pH determination, R. G. Bates, Book, Standard Methods of Chemical Analysis, 6th edition, IIIA. Instrumental Analysis, ed. F. J. Welcher, ch. 26, pp. 521–532 (D. VanNostrand Co., Inc., Princeton, N.J., 1966).

Field-ion microscopy of ruthenium at 77 °K, A. J. Melmed and R. Klein, J. Less-Common Metals **10,** 225–228 (1966).

Growth of large sodium chloride crystals from solution for color-center studies, P. M. Gruzensky, J. Chem. Phys. **43,** No. 11, 3807–3810 (Dec. 1965).

Kinetics of desorption of the beta-nitrogen states chemisorbed on tungsten, T. E. Madey and J. T. Yates, Jr., J. Chem. Phys. **44,** No. 4, 1675–1684 (Feb. 15, 1966).

Melting, flow, and thermal expansion characteristics of some dental and commercial waxes, M. Ohashi and G. C. Paffenbarger, J. Am. Dental Assoc. **72,** 1141–1150 (May 1966).

Photoionization mass spectrometry of NO, R. M. Reese and H. M. Rosenstock, J. Chem. Phys. **44,** No. 5, 2007–2009 (Mar. 1, 1966).

Recrystallization of anodic alumina films, D. J. Barber, J. Electrochem. Soc. **112,** No. 11, 1143–1145 (Nov. 1965).

Rubber and rubber products, M. Tryon and E. Horowitz, Book, Standard Methods of Chemical Analysis, 6th Edition, IIIB. Instrumental Analysis, ed. F. J. Welcher, ch. 59, pp. 1664–1763 (D. VanNorstrand Co., Inc., Princeton, N.J., 1966).

The concentration dependence of the sedimentation coefficient of polystyrene in dilute solution, I. H. Billick, Preprint American Chem. Soc. **5,** No. 2, 855–860 (1964).

The reaction between $O(^3P)$ and condensed olefins below 100 °K, A. N. Hughes, M. D. Scheer, and R. Klein, J. Chem. Phys. **70,** No. 3, 798–805 (Mar. 1966).

Titrimetric and equilibrium studies using indicators related to Nile Blue A, M. M. Davis and H. B. Hetzer, Anal. Chem. **38,** 451–461 (Mar. 1966).

ADI auxiliary publications program, R. W. Zimmerer, IEEE Trans. Microwave Theory Tech. **MTT–13,** No. 6, 883–884 (Nov. 1965).

A precision pulse-operated electronic phase shifter and frequency translator, J. Barnes and A. Wainwright, Proc. IEEE **53,** No. 12, 2143–2144 (Dec. 1965).

Can infrared improve visibility through fog? A. Ashley and C. A. Douglas, Illum. Engr. **61,** No. 4, 243–250 (Apr. 1966).

Condensate collection as a measuring technique for studying the cooling capacity of air-conditioning units, P. R. Achenbach and J. C. Davis (Proc. XI Intern. Congress of Refrigeration, Munich Germany, 1963), Book, Progress in Refrigeration Science and Technology **II,** 1281–1286 (Pergamon Press, Inc., New York, N.Y., 1963).

Cryogenic coil for megajoule energy storage, V. Arp, Proc. Intern. Symp. Magnet Technology, ed. H. Brechna and H. S. Gordon, pp. 625–629 (U.S. Atomic Energy Commission, 1965).

Design considerations for a Kossel microdiffraction camera, D. L. Vieth and H. Yakowitz, Rev. Sci. Instr. **37,** No. 2, 206–209 (Feb. 1966).

Heating performance of air-to-air heat pumps at two Air Force housing projects, P. R. Achenbach, C. W. Phillips, and W. T. Smith, ASHRAE J. **70,** 390–407 (1964).

Kinematic locator for crystal alignment, B. G. Simson and R. D. Deslattes, Rev. Sci. Instr. **37,** No. 3, 300–301 (Mar. 1966).

Laboratory study of the effect of solar radiation on cooling load of stationary refrigerated vehicles, P. R. Achenbach and C. W. Phillips (Proc. XI Intern. Congress of Refrigeration, Munich, Germany, 1963), Book, Progress in Refrigeration Science and Technology **III,** 1631–1635 (Pergamon Press Inc., New York, N.Y., 1963).

Measuring the cooling load of refrigerated vehicles by free evaporation of liquid nitrogen, P. R. Achenbach, C. W. Phillips, and R. W. Penney, Suppl. Bull. Intern. Inst. Refrigeration, Commission **VII,** 3, 115–124 (1965).

Thermodynamic property computations for systems analysis, J. G. Hust and R. B. Stewart, ASHRAE J., pp. 64–68 (Feb. 1966).

Infrared reflectance of frost, H. J. Keegan and V. R. Weidner, J. Opt. Soc. Am. Letter **56,** No. 4, 523–524 (Apr. 1966).

Lightness change of grays induced by change in reflectance of gray background, H. Takasaki, J. Opt. Soc. Am. **56,** No. 4, 504–509 (Apr. 1966).

Thoughts on weights and measures, M. W. Jensen, Soap and Chemical Specialties **VLII,** No. 4, 99 (Apr. 1966).

Analysis of the spectrum of neutral erbium (Er I), N. Spector, J. Opt. Soc. Am. **56,** No. 3, 341–349 (Mar. 1966).

Critical configurational entropy at glass transformation, S. S. Chang, A. B. Bestul, and J. A. Horman, Proc. VII Intern. Conf. Glass, Brussels, Belgium, 1965, pp. 26.1–26.15 (Gordon & Breach, New York, N.Y., 1966).

ESR measurement of metastable atomic nitrogen in helium-nitrogen afterglows, K. M. Evenson and H. E. Radford, Phys. Rev. Letters **15,** No. 24, 916–917 (Dec. 1965).

Excitation of molecular rotation by slow electrons II, S. Geltman and K. Takayanagi, Phys. Rev. **143,** No. 1, 25–30 (Mar. 4, 1966).

Field emission ultramicrometer, R. D. Young, Rev. Sci. Instr. **37,** No. 3, 275–278 (Mar. 1966).

Microscopic visualization of dose distributions, W. L. McLaughlin, Intern. J. Appl. Radiation Isotopes **17,** 85–96 (1966).

Multiple scattering corrections for proton range measurements, M. J. Berger and S. M. Seltzer, Natl. Acad. Sci.–Natl. Res. Council Publ. 1133, #5. Studies in Penetration of Charged Particles in Matter. Nuclear Science Series Report 39, pp. 69–98 (1964).

Nuclear giant quadrupole resonance, R. Ligensa, W. Greiner, and M. Danos, Phys. Rev. Letters **16,** No. 9, 364–367 (Feb. 28, 1966).

Nucleation in polymers, F. Gornick and J. D. Hoffman, Ind. Engr. Chem. **58,** No. 2, 41–53 (Feb. 1966).

Possible importance of internal excitation in ion-molecule reactions in the *F* region, L. Thomas and R. B. Norton, J. Geophys. Res. **71,** No. 1, 227–230 (Jan. 1, 1966).

Results of some recent transport calculations for electrons and bremsstrahlung, M. J. Berger and S. M. Seltzer, (Proc. 2d Symp. Protection Against Radiations in Space, Gatlinburg, Tenn., Oct. 12–14, 1964, ed. A. Reetz, Jr.), Natl. Aeronaut. Space Admin. Spec. Publ. 71, pp. 437–448 (1965).

Spin independence, W spin, parity, and SU (6) symmetry, H. J. Lipkin and S. Meshkov, Phys. Rev. **143,** No. 4, 1269–1274 (Mar. 25, 1966).

A quick, direct method for the determination of activation energy from thermogravimetric data, J. H. Flynn and L. A. Wall, Polymer Letters, **4,** Pt. B, No. 5, 323–328 (May 1966).

A standard reference material for Mossbauer spectrometry of iron and its compounds, J. J. Spijkerman, F. C. Ruegg, and J. R. DeVoe, Technical Reports Series 50, Applications of the Mossbauer Effect in Chemistry and Solid-State Physics, pp. 254–259 (International Atomic Energy Agency, Vienna, Austria, 1966).

Action of zinc dust and sodium iodide in *N,N*-dimethylformamide on contiguous, secondary sulfonyloxy groups: a simple method for introducing nonterminal unsaturation, R. S. Tipson and A. Cohen, Carbohydrate Res. **1,** No. 4, 338–340 (1965).

Approximate eigenfunctions of the Liouville operator in classical many-body systems, R. Zwanzig, Phys. Rev. **144,** No. 1, 170–177 (Apr. 8, 1966).

Crystal growth of bone mineral, W. E. Brown, Clin. Orthopaed. **44,** 205–219 (1966).

Infra-red spectra of hydroxyapatite, octacalcium phosphate and pyrolysed octacalcium phosphate, B. O. Fowler, E. C. Moreno, and W. E. Brown, Arch. Oral. Biol. **11,** 477–492 (1966).

Phase rule considerations and the solubility of tooth enamel, W. E. Brown and B. M. Wallace, Ann. N.Y. Acad. Sci. **131,** Art. 2, 690–693 (Sept. 30, 1965).

Preparation of inososes from their phenylhydrazones by use of a cation-exchange resin; separation of certain phenylhydraxones from phenylosazones, A. J. Fatiadi, Carbohydrate Res. **1,** No. 6, 489–491 (1966).

Pyrolysis of vinyl and vinylidene fluoride polymers: Influence of prior gamma-irradiation, L. A. Wall, S. Straus, and R. E. Florin, J. Polymer Sci. **4,** Pt. A–1, 349–365 (1966).

Random-walk model of adsorption of a chain-polymer molecule on a long rigid-rod molecule, R. J. Rubin, J. Chem. Phys. **44,** No. **5,** 2130–2138 (Mar. 1, 1966).

Rodlike behavior of poly(*n*-butyl)isocyanate from dielectric measurements, H. Yu, A. J. Bur, and L. J. Fetters, J. Chem. Phys. **44,** No. 7, 2568–2576 (Apr. 1, 1966).

Soft tissue response to implants of gallium alloys and silver amalgam alloys, H. W. Lyon, R. M. Waterstrat, and G. C. Paffenbarger, J. Am. Dental Assoc. **72,** No. 3, 659–664 (Mar. 1966).

A statistical model of flicker noise, J. A. Barnes and D. W. Allan, Proc. IEEE **54,** No. 2, 176–178 (Feb. 1966).

An automatic ellipsometer. Automatic polarimetry by means of an ADP polarization modulator III, H. Takasaki, Appl. Opt. **5,** No. 5, 759–764 (May 1966).

An intercomparison of atomic standards, R. Beehler, D. Halford, R. Harrach, D. Allan, D. Glaze, C. Snider, J. Barnes, R. Vessot, H. Peters, J. Vanier, L. Cutler, and L. Bodily, Proc. IEEE **54,** No, 2, 301–302 (Feb. 1966).

Bone-air cancellation, E. L. Smith, H. S. Bowman, P. G. Weissler, and R. K. Cook, Bull. Lab. Electroacoustique, No. 9, 45–50 (Apr. 1966).

Convenient safety cutoff device for water cooled equipment, J. J. Ritter and T. D. Coyle, Rev. Sci. Instr. **37,** No. 4, 523 (Apr. 1966).

Study suggests value of shared computers, A. E. Rikli, S. I. Allen, and S. N. Alexander, Mod. Hosp. **106,** No. 5, 100–108 (May 1966).

The NBS–A time scale—its generation and dissemination, J. A. Barnes, D. H. Andrews, and D. W. Allen, IEEE Trans. Instr. Meas. **IM–14,** 228–232 (Dec. 1965).

Absolute determination of refractive indices of gases at 47.7 gigahertz, A. C. Newell and R. C. Baird, J. Appl. Phys. **36,** No. 12, 3751–3759 (Dec. 1965).

International comparison of measurements at high frequencies, M. C. Selby, IEEE Spectrum **3,** No. 1, 89–98 (Jan. 1966).

LF–VLF frequency and time services of NBS, D. H. Andrews, IEEE Trans. Instr. Meas. **IM–14,** No. 4, 233–237 (Dec. 1965).

The system of electromagnetic quantities at 30 kHz to GHz, M. C. Selby, Metrologia **2,** No. 1, 37–45 (Jan. 1966).

The system of electromagnetic quantities at frequencies above 1 GHz, R. W. Beatty, Metrologia **2,** No. 1, 46–54 (Jan. 1966).

The "volt standard" moves to Gaithersburg, Maryland, W. J. Hamer, J. Wash. Acad. Sci. **56,** 101–108 (May 1966).

A nomogram for determining azimuth and horizontal trace velocity from tripartite measurements, H. Matheson, Earthquake Notes **XXXVII,** No. 1, 33–37 (Mar. 1966).

Correction in the laboratory measurement of the rate constant for $N_2^{\ddagger} + O_2 \rightarrow N_2 + O_2^{\ddagger}$ at 300 °K, F. C. Fehsenfeld, A. L. Schmeltekopf, and E. E. Ferguson, Planetary Space Sci. **13,** 919–920 (Sept. 1965).

Electromagnetic wave propagation and relativistic damping effects in a uniformly magnetized electron-positron gas, L. A. Steinert, Il Nuovo Cimento **40,** No. 10, 345–374 (Dec. 1965).

Energy gap of superconducting tin films in a magnetic field, R. S. Collier and R. A. Kamper, Phys. Rev. **143,** No. 1, 323–328 (Oct. 1966).

Extended dislocation nodes in a silver-tin alloy, L. K. Ives and A. W. Ruff, Jr., J. Appl. Phys. **37,** No. 4, 1831–1837 (Mar 15, 1966).

Force fields for group IV tetrafluorides and group V trifluorides, I. W. Levin and S. Abramowitz, J. Chem. Phys. **44,** No. 7, 2562–2567 (Apr. 1, 1966).

Longitudinal magnetoresistance of copper, R. L. Powell, Low Temp. Phys. **LT–9,** Pt. B, 732–735 (1965).

On the definition of the stacking-fault energy in binary alloys, R. deWit and R. E. Howard, Acta. Met. **14,** No. 3, 431–433 (1966).

Possible nuclear contribution to thermal expansion, E. C. Hirschkoff and N. N. Wolcott, Proc. Phys. Soc. **86,** No. 6, 1372 (1965).

Possible observation of In^{115} nuclear electric hexadecapole transi-

tions, R. J. Mahler, L. W. James, and W. H. Tantilla, Phys. Rev. Letters **16,** No. 7, 259–261 (Feb. 14, 1966).

Random-walk method for calculating correlation factors: Tracer diffusion by divacancy and impurity-vacancy pairs in cubic crystals, R. E. Howard, Phys. Rev. **144,** No. 2, 650–651 (Apr. 15, 1966).

Relaxations in polyethylene: Orientation of the lamellar crystals, R. K. Eby and J. P. Colson, J. Acoust. Soc. Am. **39,** No. 3, 506–510 (Mar. 19, 1966).

Role of surface diffusion in stabilizing the surface of a solid growing from solution or vapor, S. R. Coriell and R. L. Parker, J. Appl. Phys. **37,** No. 4, 1548–1550 (Mar. 15, 1966).

The optical constants of some oxide glasses in the strong absorption region, G. W. Cleek, Appl. Opt. **5,** No. 5, 771–775 (May 1966).

Three-body photodisintegration of He^3, H. M. Gerstenberg and J. S. O'Connell, Phys. Rev. **144,** No. 3, 834–838 (Apr. 22, 1966).

Use of ellipsometry in the study of corrosion, J. Kruger, Corrosion **22,** No. 4, 88–97 (Apr. 1966).

Wave propagation in a three-element linear spring and dashpot model filament, J. C. Smith, J. Appl. Phys. **37,** No. 4, 1697–1704 (Mar. 15, 1966).

Perturbation theorems for waveguide junction, with applications, D. M. Kerns and W. T. Grandy, IEEE Trans. Microwave Theor. Tech. **MTT-14,** No. 12, 85–92 (Feb. 1966).

The effect of atmospheric noise on the probability of error for an NCFSK system, A. M. Conda, IEEE Trans. Commun. Technol. **CT-13,** No. 3, 280–284 (Sept. 1965).

Acids, bases, and buffers, R. G. Bates, (Proc. Symp. Current Concepts of Acid-Base Measurements, New York Academy of Sciences, New York, N.Y., Nov. 23, 1964), Ann. N.Y. Acad. Sci. **133,** Art. 1, 25–33 (1966).

Comparative strengths of aliphatic acids and some other carboxylic acids in benzene at 25°, M. M. Davis and M. Paabo, J. Organic Chem. **31,** 1804–1810 (1966).

Crystal chemistry of calcium phosphates, W. E. Brown, Proc. 1964 Technical Session on Cane Sugar Refining Research, New Orleans, Louisiana, Nov. 9–10, 1964, pp. 26–34 (Agriculture Research Service, New Orleans, Louisiana, Mar. 1966).

Emission spectrometry, M. Margoshes and B. F. Scribner, Anal. Chem. **38,** No. 5, 297R–310R (Apr. 1966).

Equilibrium pressures of oxygen over oxides of lead at various temperatures, E. M. Otto, J. Electrochem. Soc. **113,** No. 6, 525–527 (June 1966).

Liquid immiscibility in the rare earth oxide-boric oxide systems, E. M. Levin, Phys. Chem. Glasses **7,** No. 3, 90–93 (June 1966).

Low-frequency motions and barrier to rotation in phosphonium iodide, J. J. Rush, J. Chem. Phys. **44,** No. 4, 1722–1723 (Feb. 15, 1966).

Matrix-isolation study of the infrared and ultraviolet spectra of the free radical CNN, D. E. Milligan and M. E. Jacox, J. Chem. Phys. **44,** No. 8, 2850–2856 (Apr. 15, 1966).

Microwave spectra of alkali hydroxides: evidence for linearity of CsOH and KOH, R. L. Kuczkowski, D. R. Lide, Jr., and L. C. Krisher, J. Chem. Phys. **44,** No. 8, 3131–3132 (Apr. 15, 1966).

Neutron and infrared spectra of $HCrO_2$ and $DCrO_2$, J. J. Rush and J. R. Ferraro, J. Chem. Phys. **44,** No. 6, 2496–2498 (Mar. 15, 1966).

Structure of cubic ammonium fluosilicate: neutron-diffraction and neutron-inelastic scattering studies, E. O. Schlemper, W. C. Hamilton, and J. J. Rush, J. Chem. Phys. **44,** No. 6, 2499–2502 (Mar. 15, 1966).

Application of sulfur coatings to integrating spheres, S. T. Dunn, Appl. Opt. **4,** No. 7, 877 (July 1965).

Simple contact type temperature controller, L. A. Marzetta, Rev. Sci. Instr. Note **37,** No. 6, 789 (June 1966).

Some statistical properties of pulses from photomultipliers, M. Gadsden, Appl. Opt. **4,** No. 11, 1446–1452 (Nov. 1965).

Use of an auxiliary sphere with a spectrophotometer to obtain absolute reflectance, D. G. Goebel, B. P. Caldwell, and H. K. Hammond, J. Opt. Soc. Am. **56,** No. 6, 783–788 (June 1966).

Anniversaries in 1965 of interest to statisticians, C. Eisenhart, Am. Stat. **19,** No. 5, 21–29 (Dec. 1965).

The assumptions underlying the analysis of variance, C. Eisenhart, (Spanish Translation by Jose Otaduy), Book, Sobre el Analisis de la Variancia, pp. 1–24 (Comision de Educacion Estadistica del Instituto Interamerican de Estadistica, Rosario, Argentina, 1953).

Debye relaxation in symmetric-top-foreign-gas mixtures: temperature dependence of collision cross sections, L. Frenkel, S. J. Kryder, and A. A. Maryott, J. Chem. Phys. **44,** No. 7, 2610–2619 (Apr. 1, 1966).

Demountable high power source for soft x-ray region, R. D. Deslattes and B. G. Simson, Rev. Sci. Instr. **37,** No. 6, 753–754 (June 1966).

EPR spectrum of gas-phase sulfur atoms, R. L. Brown, J. Chem. Phys. **44,** No. 7, 2827–2828 (Apr. 1, 1966).

Gas density stabilizer for flow proportional counters, R. D. Deslattes, B. G. Simson, and R. E. LaVilla, Rev. Sci. Instr. **37,** No. 5, 596–599 (May 1966).

Inelastic scattering cross sections for 20-keV electrons in Al, Be, and polystyrene, N. Swanson and C. J. Powell, Phys. Rev. **145,** No. 1, 195–208 (May 6, 1966).

Influence of an external magnetic field on nuclear resonance in ferromagnetic materials, L. H. Bennett, J. Appl. Phys. **37,** No. 3, 1242–1243 (Mar. 1, 1966).

Measurement of absolute wavelength stability of lasers, K. D. Mielenz, R. B. Stephens, K. E. Gillilland, and K. F. Neffien, J. Opt. Soc. Am. **56,** No. 2, 156–162 (Feb. 1966).

Measurement of the 633-nm wavelength of helium-neon lasers, K. D. Mielenz, R. B. Stephens, K. E. Gillilland, K. F. Nefflen, and R. B. Zipin, Appl. Phys. Letters **7,** No. 10, 277–279 (Nov. 1965).

Methods of determining coriolis zeta constants, S. Abramowitz and I. W. Levin, J. Mol. Spectry. **19,** No. 4, 461–462 (Apr. 1966).

Plasmon damping metals, B. W. Ninham, C. J. Powell, and N. Swanson, Phys. Rev. **145,** No. 1, 209–217 (May 6, 1966).

Polycrystalline spin wave theory of ferro-magnetic resonance compared with the tilting experiment, A. S. Risley, E. G. Johnson, and H. E. Bussey, J. Appl. Phys. **37,** No. 2, 656–668 (Feb. 1966).

Shock wave propagation in cubic lattices, D. H. Tsai and C. W. Beckett, J. Geophys. Res. **71,** No. 10, 2601–2608 (May 15, 1966).

Some applications of the wave front shearing interferometer, J. B. Saunders, Japanese J. Appl. Phys. Suppl. **I,** 4, 99–105 (1965).

Superconducting behavior of indium-lead alloys, S. Gygax, J. L. Olsen, and R. H. Kropschot (Proc. IXth Intern. Conf. Low Temperature Physics, Columbus, Ohio, Aug. 31–Sept. 4, 1964), Book, Low Temperature Physics LT9, Pt. A, ed. J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub, pp. 587–590 (Plenum Press Inc., New York, N.Y., 1965).

Superconductivity in ceramic, mixed titanates, H. P. R. Frederikse, J. F. Schooley, W. R. Thurber, E. Pfeiffer, and W. R. Hosler, Phys. Rev. Letters **16,** No. 13, 579–581 (Mar. 28, 1966).

The measurement of homogeneity of optical materials in the visible and infrared, F. W. Rosberry, Appl. Opt. **5,** No. 6, 961–966 (June 1966).

Two-state model for the free volume of vitreous B_2O_3, P. B. Macedo, W. Capps, and T. A. Litovitz, J. Chem. Phys. **44,** No. 9, 3357–3364 (May 1, 1966).

A survey of the thermodynamic data on heavier elements and their compounds: high-temperature enthalpy and vaporization equilibria, T. B. Douglas, Proc. 3d Meeting Interagency Chemical Rocket Propulsion Group, Working Group on Thermochemistry, El Segundo, Calif., Mar. 17–18, 1965, I, No. 82U, 5–16 (Chemistry Propulsion Information Agency, Silver Spring, Md., July 1965).

Activation-analysis program and facilities at the National Bureau of Standards, J. R. DeVoe and G. W. Smith, Proc. 1965 Intern. Conf. Modern Trends in Activation Analysis, Texas A & M University, College Station, Texas, pp. 225–229 (Apr. 1965).

Additional observations on the electronic spectrum of copper (II) acetate monohydrate, C. W. Reimann, G. F. Kokoszka, and G. Gordon, J. Inorg. Chem. **4,** No. 7, 1082–1084 (July 1965).

Density and specific gravity, J. K. Taylor, Encyclopedia of Industrial Analysis **1,** 546–560 (Interscience Publ. Inc., New York, N.Y., 1965).

Determination of trace amounts of tellurium in standard reference materials by neutron-activation analysis, D. A. Becker and G. W. Smith, Proc. 1965 Intern. Conf. Modern Trends in Activation Analysis, Texas A & M University, College Station, Texas, pp. 230–235 (Apr. 1965).

Electron paramagnetic resonance and nuclear magnetic resonance as analytical tools, T. C. Farrar, (Proc. Conf. Purification of Materials, New York Academy of Sciences), Ann. N.Y. Acad. Sci. **137,** 323–334 (Jan. 1966).

Equilibrium pressures of oxygen over Ag_2O-Ag at various tempera-

tures, E. M. Otto, J. Electrochem. Soc. **113,** No. 7, 643–645 (July 1966).

Force fields for some group VI hexafluorides, S. Abramowitz and I. W. Levin, J. Chem. Phys. **44,** No. 9, 3353–3356 (May 1, 1966).

Formation of crystallites of benzophenone in hydrocarbon glass, R. Keller and D. Breen, J. Chem. Phys. **43,** No. 7, 2562–2563 (Oct. 1965).

Franck-Condon factors and the mass spectra of small molecules, H. M. Rosenstock, (Proc. Intern. Mass Spectrometry Conf., ASTM E-14, The Hydrocarbon Research Group and GAMS, Paris, France, Sept. 1964), Adv. Mass Spectrometry **3,** 435–439 (1966).

Gas-phase radiolysis of cyclopentane. Relative rates of H_2-transfer reactions from various hydrocarbons to $C_3H_7^+$, R. D. Doepler and P. Ausloos, J. Chem. Phys. **44,** No. 5, 1951–1958 (Mar. 1966).

Infrared spectrum of the free radical CF_3 isolated in inert matrices, D. E. Milligan, M. E. Jacox, and J. J. Comeford, J. Chem. Phys. **44,** No. 10, 4058–4059 (May 1966).

Ion-molecule reaction in the liquid and solid phase radiolysis of hydrocarbon mixtures, P. Ausloos, A. A. Scala, and S. G. Lias, J. Am. Chem. Soc. **88,** 1583–1584 (1966).

Ion-pair process in CH_3Cl by photoionization, V. H. Dibeler and J. A. Walker, J. Chem. Phys. **43,** No. 5, 1842–1843 (Sept. 1965).

Microwave spectrum of perchlorylfluoride, D. R. Lide, Jr., J. Chem. Phys. **43,** No. 10, 3767–3768 (Nov. 1965).

Nuclear magnetic resonance studies of inorganic fluorides. II. Solvent effects on $J(^{29}Si-^{19}F)$ in silicon tetrafluoride, T. D. Coyle, R. B. Johannesen, F. E. Brinckman, and T. C. Farrar, J. Phys. Chem. **70,** No. 5. 1682–1684 (May 1966).

Phase relations between Cr_2O_3 and IrO_2 in air, C. L. McDaniel and S. J. Schneider, J. Am. Ceramic Soc. **49,** No. 5, 285–286 (May 1966).

Photoinoization of propylene at 1236 Å. Reactions of $C_3D_6^+$ with added alkanes, R. Gorden, Jr., R. Doepker, and P. Ausloos, J. Chem. Phys. **44,** No. 10, 3733–3740 (May 1966).

Progress report on a test for cleanability, M. D. Burdick, Proc. Procelain Enamel Institute Forum **26,** 41–48 (1965).

Progress report on the development of a standardized test for continuity of procelain enamel coatings, M. A. Rushmer, Proc. Porcelain Enamel Institute **27,** 49–56 (1965).

Purification of single crystal growth, J. L. Torgesen, Ann. N.Y. Acad. Sci. **137,** 30–43 (1966).

Shear relaxation times of simple fluids, R. D. Mountain and R. Zwanzig, J. Chem. Phys. **44,** No. 7, 2777–2779 (Apr. 1966).

Silicon-fluorine chemistry. II. Silicon-boron fluorides, P. L. Timms, T. O. Ehlert, J. L. Margrave, F. E. Brinckman, T. C. Farrar, and T. D. Coyle, J. Am. Chem. Soc. **87,** No. 17, 3819–3823 (Sept. 1965).

Some aspects of the coulomb hole of the ground state of H_2^-, W. Lester and M. Krauss, J. Chem. Phys. **44,** No. 1, 207–212 (Jan. 1966).

Spectrochemical analysis of high-temperature alloys by spark excitation in argon and nitrogen, H. C. Dilworth, Am. Soc. Testing Mater. Spec. Tech. Publ. 376 (1965).

Standardization of the differential chemical shift for Fe^{57}, J. J. Spijkerman, F. C. Ruegg, and J. R. DeVoe (Proc. New England Nuclear Mossbauer Symp., New York, N.Y., Jan. 1964), Book, Mossbauer Effect Methodology, pp. 115–120 (Plenum Press Inc., New York, N.Y., 1965).

The electron paramagnetic resonance spectrum of tetrakis-*t*-butoxyvanadium(IV), G. F. Kokoszka, H. C. Allen, Jr., and G. Gordon, Inorg. Chem. **5,** 91–93 (1966).

The microwave spectrum of dinitrogen trioxide, R. L. Kuczkowski, J. Am. Chem. Soc. **87,** 5259–5260 (1965).

A cooling system for a laboratory magnet, R. A. Forman and T. McKneely, Appl. Spectry. **20,** 189 (May–June 1966).

A large signal IGFET DC source follower, D. P. Stokesberry, Proc. IEEE **54,** No. 1, 66 (Jan. 1966).

A method for calibrating volt boxes, with analysis of volt-box self-heating characteristics, R. F. Dziuba and T. M. Souders, IEEE Intern. Conf. Record, Pt. 10, pp. 17 (Mar. 21–25, 1966).

A semi-automatic technique for tuning a reflectometer, M. H. Zanboorie, IEEE Trans. Microwave Theory Tech. **MTT–13,** No. 5, 709–710 (Sept. 1965).

A simple current generator, J. K. Whittaker, Nuclear Instr. Methods **39,** No. 1, 183–184 (Jan. 1966).

A zero-crossing discriminator with picosecond time slewing, J. K. Whittaker, IEEE Trans. Nuclear Sci. **NS–13,** No. 1, 399–405 (Feb. 1966).

Anomalous conductance behavior in polymers, A. H. Scott (1965 Annual Report Conf. Electrical Insulation), NAS–NRC Publ. 1356, pp. 98–102 (Natl. Acad. Sci.–Natl. Res. Council, Washington, D.C., 1966).

Apparent negative impedances and their effect on three-terminal dielectric loss measurements, W. P. Harris (1965 Annual Report Conf. Electrical Insulation), NAS–NRC Publ. 1356, pp. 83–86 (Natl. Acad. Sci.–Natl. Res. Council, Washington, D.C., 1966).

Effects of thermal shrinkage on built-up roofing, W. C. Cullen, J. Bldg. Res. Inst. **2,** No. 4, 23 (July–Aug. 1965).

Fire test methods for paints, A. F. Robertson, Bldg. Res. pp. 48–52 (July–Aug. 1965).

Flow measurement standardization, T. Filban and M. R. Shafer, Proc. ISA Conf. 1964, 19, Pt. 1, Paper No. 12–2–4–64 (1964).

Guide to dental materials, 1966–1967, G. C. Paffenbarger, J. W. Stanford, M. P. Kumpula, and W. T. Sweeney, Am. Dental Assoc. 3d ed., 188 pages (Chicago, Ill., 1966).

Human engineering with design of a console for the comparison of volt boxes, P. H. Lowrie, Jr., ISA J. pp. 67–71 (July 1965).

International intercomparison of standards for microwave power measurement, G. F. Engen, IEEE Trans. Microwave Theory Tech. **MTT–13,** No. 5, 713–715 (Sept. 1965).

Measuring the coordinates of a number of points in a complex laboratory-scale model, F. D. Ordway, Rev. Sci. Instr. **36,** No. 8, 1156–1159 (Aug. 1965).

Modular software for on-line handling of nuclear data, J. M. Wyckoff, IEEE Trans. Nuclear Sci. **NS–13,** No. 1, 199 (Feb. 1966).

National Bureau of Standards offers method to prevent built-up roof splitting failure, W. C. Cullen, Roofing Siding Insulation Mag. **40,** No. 4, 28–30 (Apr. 1965).

1965 Meeting of IEC Committee TC45, Progress in Nuclear Instrumentation, L. Costrell, Mag. Std., p. 9 (Jan. 1966).

Roofing research in Guam and Okinawa, W. C. Cullen, Am. Roofer Bldg. Improvement Contractor, pp. 10 (Oct. 1965).

Standards for galvanomagnetic devices, S. Rubin, Solid State Electronics **9,** 5, 559–566 (May 1966).

Suppression of 60 c/s pickup when using sensitive DC amplifiers, R. L. Nuttall and D. C. Ginnings, Electron. Design 12, No. 26, pp. 34–35 (Dec. 20, 1965).

Tapered inlets for pipe culverts, discussion and author's closure, J. L. French, J. Hydraulics (Div. Am. Soc. Civil Eng.) pp. 286 (May 1965).

The effect of insulation on the durability of a smooth-surfaced built-up roof, W. C. Cullen and W. H. Appleton, Coal Tar Asphalts and Related Mater. **3,** 122–124 (Mar. 1965).

The NBS instrumentation for use in the measurement of spectral irradiances of solar simulators, R. Stair and W. E. Schneider, Proc. Intern. Symp. Solar Radiation Simulation, Los Angeles, Calif., Jan. 18–20, 1965, pp. 75–94 (Inst. Environmental Sciences and Am. Soc. Testing and Materials, 1965).

The response of flush diaphragm pressure transducers to thermal gradients, L. Horn, ISA 20th Annual Conf. and Exhibit, Los Angeles, Calif., Oct. 4–7, 1965, 20, Pt. III, ISA Preprint No. 13.3–4–65 (1965).

Thermal insulation in buildings, F. J. Powell, Military Engr. **58,** No. 38, 275–276 (July–Aug. 1966).

Voltmeter calibration to 1 GHz, M. C. Selby, W. J. Blank, and R. P. Chairton, (Proc. 1965 Wescon Conf., San Francisco, Calif., Aug. 1965), IEEE WESCON Tech. Papers **9,** Pt. 6, No. 8.8, 1–12 (1965).

The design of a federal statistical data center, D. Rosenblatt, E. Glaser, and M. K. Wood, Statistical Evaluation Report No. 6, Office of Statistical Standards, Bureau of the Budget, Appendix C, pp. 1–19 (Office of the President, Washington, D.C., Dec. 1965).

Color designation and specification, D. B. Judd, Encyclopedia of Industrial Analysis **1,** 315 (1966).

Electrical testing in chemical technology, A. H. Scott, Encyclopedia of Chemical Technology, 2d ed., **7,** 716–726 (1965).

Measurement of DC dielectric conductance (reciprocal resistance) at elevated temperatures, A. H. Scott, Proc. 6th Electrical Insulation Conf., Sept. 13–16, 1965, pp. 252–254 (1965).

Metrology essential to the space age, A. G. McNish, George Washington University Mag. **1,** No. 4, 10–14 (1965).

Progress report for OSA Committee on uniform color scales, D. B. Judd, Proc. Intern. Color Comm., Lucerne, Switzerland, 1965, 1, 399–407 (Mar. 1966).

Specification and designation of color, D. B. Judd and I. Nimeroff, Treatise Anal. Chem. **36,** 2873–2942 (1964).

The calibration of permanent magnet standards, I. L. Cooter, ISA 20th Annual Conf. and Exhibit, Part 1. Measurement Standards and Instrumentation, Los Angeles, Calif., Oct. 4–7, 1965, 20, Pt. 1, ISA Preprint 14.1–3–65 (1965).

The international system of units, A. G. McNish, American Society for Quality Control Annual 1965 Tech. Conf. Trans. Industrial Quality Control **22,** No. 9, 465–469 (Mar. 1966).

The nature of measurement, A. G. McNish, Book, Industrial Metrology (Sponsored by the American Society of Tool and Manufacturing Engineers), ch. 1, 4, No. 2, 24–31 (Mar. 1966).

Color appearance, D. B. Judd, Proc. Intern. Color Comm., Lucerne, Switzerland, 1965, 1, 27–51 (Mar. 1966).

Coolers for semiconductor light emitters, lasers and photodetectors, H. K. Kessler, Rev. Sci. Instr. **37,** 517–518 (Apr. 1966).

Decays of positive-parity baryon resonances in a broken U(12), H. Harari, D. Horn, M. Kugler, H. J. Lipkin, and S. Meshkov, Phys. Rev. **140,** No. 4B, B1003 (Nov. 1965).

Discussion of ASME paper 60–WA–333, November 30, 1960, D. P. Johnson and U. O. Hutton, Book, High Pressure Measurement, ed. A. A. Giardini and E. C. Lloyd, pp. 350–351 (Butterworth, Inc., London, England, 1963).

Effects of space radiation on refractive properties of optical glass, I. H. Malitson, M. J. Dodge, and M. E. Gonshery, Proc. Annual Conf. Photography, Science and Engineering, San Francisco, Calif., May 9–13, 1966, p. 75 (1966).

Elastic constants of single crystal UO_2 at 25° C, J. B. Wachtman, Jr., M. L. Wheat, H. J. Anderson, and J. L. Bates, J. Nuclear Mater. **16,** 39–41 (1965).

Electron energy bands in $SrTiO_3$ and TiO_2 (theory and experiment), A. K. Kahn, H. P. R. Frederikse, and J. H. Beckerd, (Informal Proc. Bull., Intern. Conf. Materials, Pittsburgh, Pa., 1963), Book, Transition of Metal Compounds, ed. E. R. Schatz, pp. 53–64 (Gordon and Breach Sci. Publ., New York, N.Y., 1964).

Fundamental studies of color vision from 1860 to 1960, D. B. Judd, Proc. Natl. Acad. Sci. **55,** No. 6, 1313–1330 (June 1966).

Geomagnetic and solar data, J. V. Lincoln, J. Geophys. Res. **70,** No. 13, 3227–3228 (July 1965); 70, No. 15, 3761–3763 (Aug. 1965); 70, No. 17, 4383–4384 (Sept. 1965); **70,** No. 19, 4963–4964 (Oct. 1965)

High resolution infrared spectra of cyanogen and cyanogen-$^{15}N_2$, A. G. Maki, J. Chem. Phys. **43,** No. 9, 3193–3199 (Nov. 1965).

Interspecimen comparison of the refractive index of fused silica, I. H. Malitson, J. Opt. Soc. Am. **55,** No. 10, 1205 (Oct. 1965).

Intrinsic and lattice-induced distortion of the tetrachlorocuptrate ion, M. Sharnoff and C. W. Reimann, J. Chem. Phys. **43,** No. 9, 2993–2996 (Nov. 1965).

Line broadening, W. L. Wiese, Book, Plasma Diagnostic Techniques, ed. R. H. Huddleston and S. L. Leonard, 21, 265–317 (Academic Press Inc., New York, N.Y., 1965).

Magnetic susceptibility of insulating and semiconducting strontium titanate, H. P. R. Frederikse and G. A. Candela, Phys. Rev. **147,** No. 2, 583–584 (July 15, 1966).

Magnetoacoustic adsorption and the fermi surface in potassium, H. J. Foster, P. H. Meijer, and E. V. Mielczarek, Phys. Rev. **139,** No. 6A, A1849–A1857 (Sept. 1965).

Magnetoresistance of semiconducting $SrTiO_3$, H. P. Frederikse, W. R. Hosler, and W. Thurber, Phys. Rev. **143,** No. 2, 648–651 (Mar. 1966).

Measurements of ion-optical properties of a high resolution spectrometer for electron scattering, S. Penner and J. W. Lightbody, Proc. Intern. Symp. Magnetic Technologies, Stanford Linear Accelerator Center, Stanford Univ., Calif., Sept. 8–10, 1965, pp. 154–163 (Oct. 1965).

Newly observed structure in the photoionization continua of Kr and Xe below 160 Å, K. Codling and R. P. Madden, Appl. Opt. 4, 1431–1434 (Nov. 1965).

Observations of dislocations in ammonium dihydrogen phosphate: production of dislocation-free crystals, R. D. Deslattes, J. L. Torgesen, B. Parketzkin, and A. T. Horton, J. Appl. Phys. **37,** No. 2, 541–548 (Feb. 1966).

Observatory reports, spectroscopy, C. E. Moore, Astron. J. **70,** No. 9, 637–639 (Nov. 1965).

Optical absorption of cobalt in manganese fluoride, R. F. Blunt, J. Chem. Phys. **44,** No. 6, 2317–2320 (Mar. 15, 1966).

Photon absorption cross section of spherical nuclei, M. G. Huber, H. J. Weber, M. Danos, and W. Greiner, Phys. Rev. Letters **15,** No. 12, 529–531 (Sept. 1965).

Photoneutron reactions, E. G. Fuller (Proc. Intern. Conf. Study of Nuclear Structure with Neutrons, Antwerp, Belgium, July 14–23, 1965), Book, Nuclear Structure Study with Neutrons, ed. M. Neve De Mevergnies, P. VanAssche, and J. Vervier, pp. 359–375 (North-Holland Publ. Co., Amsterdam, The Netherlands, 1966).

Photonuclear reactions, M. Danos and E. G. Fuller, Ann. Rev. Nuclear Sci. **15,** 29–66 (1966).

Radiative recombination lifetimes in laser excited silicon, N. N. Winogradoff and H. K. Kessler, Appl. Phys. Letters **8,** No. 4, 99 (Feb. 1966).

Radiation of sound by earthquakes, R. K. Cook, 5e Cong. Intern. Acoustique, Liege, Belgium 1b, K19 (1965).

Relation of strength of thermally tempered glass to fracture mirror size, M. K. Kerper and T. G. Scuderi, Am. Ceramic Soc. Bull. **44,** No. 12, 953–955 (Dec. 1965).

Resolving power related to aberration, F. E. Washer, Photogrammetric Eng. **XXXII,** No. 2, 213–226 (Mar. 1966).

Second breakdown and current distributions in transistors, H. A. Schafft and J. C. French, Solid-State Electron. **9,** No. 7, 681–688 (July 1966).

Secondary electron trajectories in a Faraday cup magnetic field. J. S. Pruitt, Nuclear Instr. Methods **39,** 329 (Feb. 1966).

Structure in the photo-ionization continuum of N_2 near 500 Å, K. Codling, Astrophys. J. **143,** No. 2, 552–558 (Feb. 1966).

Symmetry conditions on jump rates occuring in relaxation times associated with point defect motion, H. S. Peiser and J. B. Wachtman, Jr., J. Phys. Chem. Solids **27,** 975 (1966).

Tables of dielectric constants, dipole moments and dielectric relaxation times, F. I. Mopsik, (1964 Digest of Literature on Dielectric Conf. Electrical Insulation), NAS–NRC Publ. 1342, 28, 33–66 (Natl. Acad. Sci.-Natl. Res. Council, Washington, D. C., 1965).

The interaction of radiation with charged particles. I, W. H. Chappell, W. E. Brittin, and S. J. Glass, Il Nuovo Cimento **38,** No. 3, 1186–1191 (1965).

The interaction of radiation with charged particles. II. Dispersion relation for transverse modes, S. J. Glass and W. R. Chappel, Il Nuovo Cimento **38,** No. 1, 79–88 (1966).

The paramagnetic resonance spectrum of W^{5+} in rutile (TiO_2), T. T. Chang, Phys. Rev. **147,** No. 1, 264–267 (July 8, 1966).

Atmospheric research and electromagnetic telecommunication, Part I, H. G. Booker and C. G. Little, IEEE Spectrum **2,** No. 8, 44–52 (Aug. 1965).

Atmospheric research and electromagnetic telecommunication, Part II, H. G. Booker and C. G. Little, IEEE Spectrum **2,** No. 9, 98–103 (Sept. 1965).

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Library of Congress Catalog Card Number: 62-4414

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Price 75 cents (single copy). Subscription price: $2.25 a year, $0.50 additional for foreign mailing.
UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1966